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TIDAL COMPUTATIONS IN SHALLOW WATER

J. J. DRONKERS

J. C. SCHÖNFELD

REPORT ON HYDROSTATIC LEVELLING ACROSS THE WESTERSCHELDE

A. WAALEWIJN

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The views in this report are the authors' own

FOREWORD

The Netherlands governmental civil engineers organisation, called Rijkswaterstaat, has the pleasure of presenting this the first number of a new series which describes the mathematical methods now in use to calculate the changes of tides and tidal or non-tidal currents during and after the execution of works which will influence them. The experience gained since 1920, when Prof. H. A. Lorentz started his well-known mathematical prediction about the influence of the Zuiderzee dam upon the tides and storm-surges, has steadily increased. His method has been used extensively since then and various other mathematical methods have been added, so that there are now a few of them which can check each other. From these the expert may choose the method most suitable for the problem under review.

The Delta works (1953 till about 1980) require very precise tidal calculations based on equally precise data obtained from the gauges. For this reason the basic data obtained from the gauges should be quite accurate, especially in the southwestern part of the Netherlands, where the new works have started, which will close all the estuaries except the Westerschelde and the Rotterdam Waterway. The first is too wide to be able to be crossed by optic levelling, hence the necessity of the hydraulic levelling across this estuary, also described in this volume.

It is my sincere wish that the new series, to be issued at irregular intervals, may render good service in the international field of civil engineering.

The Director-General,

A. G. Maris.

TIDAL COMPUTATIONS IN SHALLOW WATER

J. J. DRONKERS [1])
J. C. SCHÖNFELD [2])

SUMMARY

A survey is given of the established practice of tidal computations in the Netherlands.

The dynamical analysis of tidal elevations and currents in shallow waters is of great practical interest when coastal engineering projects are undertaken, like the Zuiderzee or the Delta project, both in the Netherlands.

Tidal computations have to be based upon a careful schematization of the region considered. The water is assumed to move substantially in the length direction of a channel with variable cross-section or in a channel network. The problem is then governed by two partial differential equations, the continuity equation and one dynamical equation.

These equations generally are too complicated to admit solutions in closed form. Several methods of numerical approach have therefore been devised.

First among these methods are the harmonic methods, by which one or more tidal constituents are computed. If confined to one single constituent (M2), this method is relatively simple to handle and it requires a moderate effort of calculation. The computation of further constituents leads to progressively increasing efforts, so that the method is seldom extended beyond the second harmonic (M4).

For the immediate numerical integration of the continuity and the dynamical equation, several ways may be followed. A method based on an iterative process has been extensively used for practical problems in the Netherlands and abroad. This method is particularly suited to analyse tidal motions in connection with the problem of schematization.

A third group of methods is based upon the properties of the characteristic elements of the differential equations. Integration, performed either graphically or numerically, is in particular used in specific propagation problems, such as wave motions produced by sluicing operations and bores.

The employment of large computers, either analogue or digital, is here mentioned only briefly, since more detailed information on the development is being prepared.

SOMMAIRE

La pratique établie des calculations de marée en Pays Bas est résumée.

L'analyse dynamique des marées et des courants de marée en profondeur faible a beaucoup d'intérêt pratique pour l'exécution des ouvrages dans les régions littorales, comme les projets du Zuiderzee et du Delta dans les Pays Bas.

La calculation des marées doit partir d'une schématisation scrupuleuse de la région en considération. Il est supposé que le mouvement soit à peu près longitudinal dans un chenal à profil varié, ou dans un réseaux de chenaux. Le problème se pose alors par deux équations aux dérivées partielles, l'équation de continuité et une équation dynamique.

Ces équations sont généralement trop compliquées qu'ils admisent une solution en forme fermée. C'est pour ça que diverses méthodes numériques ont été inventées.

Premièrement il faut mentionner les méthodes harmoniques par lesquelles on calcule une ou plusieurs composantes sinusoidales. En se bornant à une seule composante (M2) on obtient une méthode relativement simple et n'exigeant qu'un effort de calcul modéré. En calculant plus de

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composantes l'effort va en croissant progressivement, tel que la méthode est rarement étendue au delà de la seconde composante (M4).

L'intégration numérique directe de l'équation de continuité et de l'équation dynamique se peut faire par diverses méthodes. Une méthode basée sur une procédure itérative a été employée intensivement pour des problèmes pratiques aux Pays Bas et d'ailleurs. Cette méthode se prête surtout pour l'analyse des mouvements de marée en connection avec la problème de la schématisation.

Les méthodes d'une troisième groupe se basent sur les éléments caractéristiques des équations différentielles. L'intégration, soit graphiquement, soit numériquement, s'emploie surtout pour des problèmes spéciaux de propagation, comme les mouvements ondulatoires produits par les opérations de vannage et les mascarets.

L'emploi de grands calculateurs, soit analogues, soit digitales, est mentionné ici que brèvement, parce qu'un rapport plus détaillé sur ce développement est en cours de préparation.

1. INTRODUCTION

1, 1. Purposes of computation.

Hydraulic engineering in maritime waters is confronted with the tidal motion. It depends on the extent of the structures to be planned, how deeply the engineer will be interested in the tides.

When he is concerned with local structures of relatively small extent, so that there will be no serious interference with the tidal motion as a whole, it will be sufficient to collect observational data on the tide as it exists. The influence of the structure on the local pattern can then usually best be investigated by a model of the local situation.

If however a substantial interference with the movement of the tides is contemplated, it will be necessary to investigate thoroughly the mechanism of the tidal motion, in order to predict correctly how the intended interference will work out. For this purpose the engineer can have recourse to computations and to research on a model of the whole estuary.

Technical projects which may entail such a thorough investigation are e.g.:

1. Land reclamation in an estuary.
2. Safeguarding low countries along an estuary from flooding by storm surges.
3. Improvement of draining of low countries along an estuary.
4. Preventing or impeding the intrusion of salt water through an estuary.
5. Preventing the attack of a tidal current on a shore.
6. Improvement of a shipping channel in an estuary.
7. The construction of a shipping canal in open connection with the sea.
8. The utilization of the energy of the tides.

Usually a technical project will cover more than one of the above purposes.

The computations provide informations on water levels (of interest for height of seawalls, draining sluices, draught of ships), currents (shore protection, silting up or deepening of channels, navigation) and energy (power plant). They give indications for the execution of works, in particular for closing programs of stream gaps. Sometimes computations may have influence on the general design of a project, like in the case of the enclosure of the Zuiderzee [12], where the location of the main dam was altered according to insight gained by computations. None of the many works in the Dutch tidal waters executed since, was undertaken without the support of tidal computations.

1, 2. Nature of the problem.

Tidal hydraulics in shallow water deals with the mechanism of the tidal motion in estuaries, inlets, tidal rivers, open canals, lagoons and other coastal waters. For brevity we shall hereafter often speak of "estuaries", when we mean those shallow coastal waters in general.

The astronomical tides are generated substantially in the vast oceans and thence penetrate directly or through border seas into the coastal waters just mentioned. The tidal motion in this final stage may be characterized by the following properties:

1. On the whole the tides belong to the kind of wave phenomena called "long waves", i.e. waves in which the vertical velocities and accelerations are negligible. Only the tidal phenomenon known as "bore" forms an exception (cf 5, 4).
2. The estuaries are usually so shallow that various effects which are almost imperceptible in deep water, such as bed friction and nonlinear distortion, become appreciable or even predominant.
3. Shores and shoals substantially impose the direction of the flow of water. The estuary may therefore be considered theoretically as a channel or as a network of channels.
4. The inlet in which the tides penetrate is usually so much more narrow and shallow than the sea or ocean whence the tides come, that the reaction by the inlet on the sea or ocean is negligible or at most a secondary effect. Hence the tidal motion at the offing of the inlet may be considered as the given source of the motion in the estuary.

The tides are seldom of purely astronomical origin. They are in particular often affected by meteorological conditions (storm), sometimes in a considerable degree. Since in shallow water the nonlinear effects are strong, the deviations from the astronomical tide (the storm surge) can not be well considered separately. On the contrary the composite motion resulting from the combined astronomical and meteorological forces must be treated as one integral phenomenon. Such tides affected by storms will hereafter be called "storm tides".

Tidal hydraulics deals with storm tides as well as with the normal undisturbed tides.

The tidal motion in a channel can be described by two differential equations, the one expressing the conservation of mass (equation of continuity), and the other expressing the equilibrium of forces and momentum in the length direction of the channel (dynamical equation). The ways of dealing mathematically with these equations can be grouped as follows:

1. *Harmonic methods.* The composite tidal motion is resolved into harmonic components by Fourier series, and these harmonic components are treated separately while terms for their mutual interaction are introduced.

2. *Direct methods.* The equations are subjected immediately to some process of numerical integration, e.g. by an iterative process, power series expansions, or by converting the differential equations into equations of finite differences.

3. *Characteristic methods.* The propagation of the tidal waves is analysed on the basis of the theory of the characteristic elements of the differential equations.

1, 3. Historical survey.

At the beginning of the development of tidal hydraulics we meet the work in England by Airy [1], which dates from the first half of the 19th century. Airy treated the tides as periodic waves which he resolved into harmonic components. He demonstrated that, by the nonlinear character of the propagation, an originally purely sinusoidal wave is distorted in such a way that higher harmonic components are being introduced.

After the middle of the 19th century de Saint Venant [2] in France approached the propagation of tidal and similar long waves from another side. Although the theory of the characteristics is not explicitly mentioned, it yet forms the mathematical background of de Saint Venant's work. A contribution in this field was given likewise by MacCowan [3] in England.

Full emphasis on the value of the characteristics for defining the propagation of tides is laid by the Belgian Massau [5]. His work, which dates from 1900, has attracted less attention from tidal hydraulicians than it deserved. Its merits have only been fully understood about half a century afterwards.

In the 20th century the question of practically computing the tidal movement in an estuary comes to the fore. De Vries Broekman [7] (Netherlands) was the first to point out the possibility of such a computation by a direct method of finite differences, and Reineke [9] likewise developed a direct method and applied it to German rivers.

The art of tidal computations received great stimulus by the decision to partially enclose the large estuary of the Zuiderzee in the Netherlands. A state committee under the presidence of the great physicist Lorentz was entrusted with the investigation of the tidal problems of the Zuiderzee. [12]. The committee followed two ways of approach.

Firstly Lorentz contrived by an ingenious artifice to linearize the quadratic resistance in such a way, that the fundamental harmonic component is rendered with great accuracy. On this basis a computation method was developed to determine the M2 component of the normal tidal movement in the channel network of the Zuiderzee [11]. The method was used to predict the modifications in the tides after the enclosure.

Secondly a direct method by power series expansion was developed by which some computations of storm tides were performed.

The work of the Lorentz committee proved to be a fertile ground for the further development. The quadratic character of the frictional resistance encountered by a tidal flow had always been one of the main practical difficulties for computations. Although Lévy (France) at the end of the 18th century already had put forward the principle of linearization in computing the tide penetrating up a river [4] and Parsons (U.S.A.) had given a treatment by linearized equations in his study of the Cape Cod canal [8] [1]), the real clue has been the principle of Lorentz. The extension of this principle to rivers with a fluvial discharge was taken up by Mazure who developed a method to compute the M2 tidal component on the Dutch rivers [17].

The next step in the Netherlands was the analysis of other harmonics as done by Airy, but with the frictional resistance taken into consideration; Dronkers [21], Stroband [20] and Schönfeld [28] have each contributed to the solution of this problem.

The work of Van Veen [16] may likwise be mentioned in this context, although it bears not so much on computation methods as on the technique of the electric analogue of a tidal system.

The direct method by power series of the committee Lorentz was made fit for tidal rivers by Dronkers [14]. In a later stage the power series were converted into expansions by an iterative process [21, 24]. Many tidal problems have been analysed more or less intensively by these methods in the course of years [27, 32].

In the post-war period Holsters [19] (Belgium) re-discovered the work of his compatriot Massau. The method of cross-differences which he developed and presented by the name "method of lines of influence" as an approximate characteristic method, should in fact be classified as a direct method. [33] (cf 4, 1).

The method presented by Lamoen [25] (Belgium) is an approximate characteristic method in which the nonlinear features of the propagation are neglected, but in which the frictional resistance is computed correctly.

A more refined application of the theory of the characteristics was given by Schönfeld [28] who demonstrated the value of the characteristic analysis for the fundamental discussion of the propagation of the tides.

The paper deals with its subject as follows:

First the mathematical formulation of a tidal problem is discussed (Ch. 2).

Next the groups of methods of computation are expounded in chronological order (Chs. 3, 4, 5). In each chapter the most simple method of the group is treated in order to demonstrate the principle. Then the more refined methods follow.

Finally a comparative discussion of the methods of computation is given (Ch. 6). The fields of their application in European, and more particularly Dutch practice, are indicated. Moreover a comparison with model research is made.

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1) A more recent American publication is Pillsbury's "Tidal Hydraulics" (1938), which we must leave out of the discussion to our regret, as we have not been able to lay hands on a copy.

List of basic symbols

A	cross-sectional area of streambed	
$a\,(a_s)$	depth below water surface	$a_s = A/b_s$
B	storing area of a section	
b	storing width of water surface	$b = B/l$
b_s	surface width of streambed	$b_s = \delta A/\delta h$
C	Chézy coefficient of flow	
$c\,(c_0, c^+, c^-)$	velocity of propagation	
F, G	characteristic wave components	
g	gradient of gravity	
H	total head above datum	$H = h + v^2/2g$
h	water level above datum	
$i\,(i_r, i_s, i_a)$	inclination (= slope)	
j	imaginary unit	$j^2 = -1$
K	conveyance of cross-section of streambed	$K = CA\sqrt{a_s}$
l	length of section	
M	inertance of section	$M = lm$
m	inertance per unit length	$m = 1/gA$
Q	discharge (ebb positive)	
$q\,(q_i)$	discharge per unit width (length)	
R	linearized resistance of section	$R = lr$
r	linearized resistance per unit length	
t	time	
U	kinetic factor	$U = 1/2gA^2$
v	velocity of flow	$v = Q/A$
W	quadratic resistance of section	$W = lw$
w	quadratic resistance per unit length	$w = 1/K^2$
x	coordinate along channel (positive in seaward sense)	
$Z_0\,(Y_0)$	characteristic wave impediment (wave admission)	
ρ	density	
$\tau\,(\tau_0)$	time of propagation of section	$\tau = l/c$
$\varnothing_n$	relative phase angle of n-th harmonic tides	$\varnothing_n = \arg - \mathbf{Q}_n/\mathbf{H}_n$
ω	angular frequency of fundamental tide	
$\mathbf{H}\,(\mathbf{Q})$	complex amplitude of vertical (horizontal) harmonic tide	
$\mathbf{Z}\,(\mathbf{Y})$	complex tidal impedance (admittance)	
$\mathbf{Y}_p\,(\mathbf{y}_p)$	parallel admittance, of section (per unit length)	
$\mathbf{Z}_s\,(\mathbf{z}_s)$	series impedance, of section (per unit length)	
$\mathbf{K}\,(\mathbf{k})$	complex propagation exponent, of section (per unit length)	
$\mathbf{H}_n\,(\mathbf{Q}_n)$	complex n-th harmonic Fourier coefficient of	$H\,(Q)$

2. THE BASIS OF TIDAL COMPUTATIONS

2, 1. Schematization of an estuary.

Most tidal waters have an irregular shape as well in plan as in longitudinal or transversal section. Every irregularity like a shoal, isle, groyne etc., has its influence on the local pattern of flow. It would be a considerable complication of computations if all these local situations had to be considered in detail. Fortunately this is generally not necessary since it is possible to compute a tidal motion accurately by means of a rather severely schematized mathematical model, provided this model represents correctly some particular condensed characteristics of the esturay. This must be checked if possible by analysing well observed tides.

We confine ourselves here to the case of an estuary or other tidal water with such a small width compared to the wave length of the tide (cf 3, 1), that the tidal flow is directed mainly in the length of the estuary.

The bed of the estuary fulfils two hydraulic functions. Firstly the bed conveys the flow of water in the length direction. Secondly the bed stores quantities of water as the tide rises and returns them during the falling tide. Not all the parts of the bed necessarily partake to the same degree in the two functions. Parts of the bed (the channels) partake in both functions. Other parts however (like shoals, compartments between groynes, dead branches, flooded areas, harbour areas) contribute appreciably to the storing function but not or relatively little to the conveying function.

The estuary is now schematized as a channel that conveys and stores, the streambed, and adjacent to it regions that store but do not convey. The velocity distribution in the streambed is assumed to be uniform.

The boundary between the streambed and the adjacent storing regions is sometimes well defined by the actual situation, e.g. in a river with a dead branch. In other circumstances, when there is in fact a gradual transition so that the boundary is fictitious, the schematization is nonetheless justified, provided the dimensions of the streambed are defined appropriately. Although this can be rationalized, it always remains for a good deal a matter of experience.

The conveying cross-section varies with the water level, not only because the depth varies but also because there may be parts of the bed such as shoals, that are contributing to the conveying function when the level is high, but not when it is low.

If the cross-section of the streambed is further schematized by a rectangle, it may therefore be necessary to apply different schematizations for high and low levels. This may entail that a storm tide is computed with another schematization than an ordinary tide (cf fig. 1).

A second schematization is necessary in view of the variation of the cross-sections along the conduit. For that reason the conduit is divided into sections of not too great length. In each section an average cross-section of the streambed is defined and the streambed in the section is treated as a prismatic channel with that average cross-section.

The total storing area B of the water surface in the section, which is a function of the level h, is divided by the length of the section and this quotient is considered as

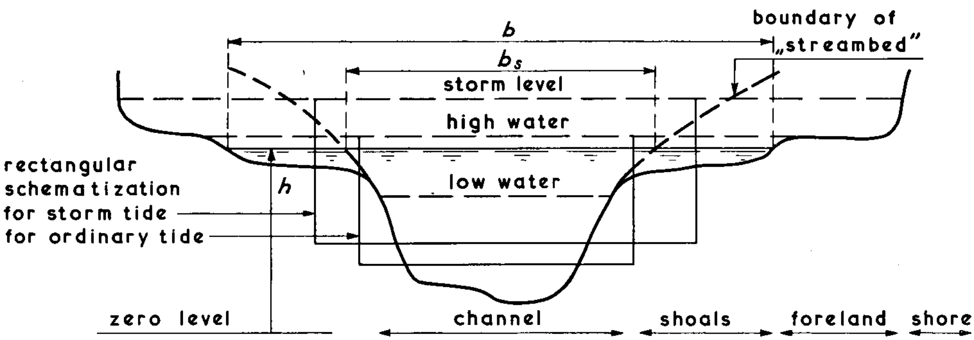

Fig. 1. Cross-section of an estuary.

the storing width b of the section. This is generally different from the surface width b_s of the streambed ($b \geqslant b_s$).

Most tidal currents encounter appreciable losses of head by dissipative forces. As a rule the losses by friction along the bottom are predominant but there may also be appreciable losses by curvature of the channel, by its widening and narrowing and by obstacles like groynes, bridge piers etc. Provided the sections are not too long, it is tolerable to merge all losses of a section into an equivalent frictional loss of head distributed uniformly along the section.

When an estuary is formed by a network of channels, the shoals between the channels may be divided into separate storing regions of the adjacent channels, if necessary with a correction for transmission of water over the shoals from one channel to another.

A complete analysis of the estuary must deal with the channel network in all its details. For more restricted purposes however, parallel channels may be schematized by replacing them by one channel with a composed cross-section.

The admissable length of the sections depends partly on the regularity of the estuary and partly on the character of the tidal motion. In a very regular conduit the sections may be longer than in a very irregular one. Even in a very irregular channel, however, it is sufficient that the length of the sections be small compared to the wave length of the tide. In Dutch tidal practice sections of 5 to 10 km are used as a rule.

2, 2. **The differential equations.**

The tidal motion in the length direction of an estuary is mathematically described by two differential equations. They can be derived by considering mass and momentum.

Continuity equation. This follows from the law of conservation of mass:

(201)
$$\frac{\delta Q}{\delta x} + b\frac{\delta h}{\delta t} = 0.$$

It expresses that the difference in discharge between cross-sections x and $x + dx$, and the accumulation or evacuation of water by rising or falling of the level, must balance each other.

When a supplementary discharge per unit length q_i, e.g. to an area that is being flooded over a dyke, must be accounted for, we have

(202)
$$\frac{\delta Q}{\delta x} + b\frac{\delta h}{\delta t} + q_i = 0,$$

where the term q_i may depend on the head of water.

Dynamical equation. This is based on Newton's law, which is not easily applied directly since the mass of water we consider is variable by the transport between the streambed and the adjacent storing regions. For this reason it is more convenient to use the law of conservation and variation of momentum per unit length and in the longitudinal direction of the estuary. From this equation we substract ρv times (201) and after introduction of some approximations of minor importance, we arrive at the dynamical equation

(203)
$$\frac{\delta H}{\delta x} + \frac{1}{g}\frac{\delta v}{\delta t} - (1 - \gamma)\, v\,\frac{b - b_s}{gA}\,\frac{\delta h}{\delta t} + i_r = 0.$$

The first term represents the gradient of the total head and the second term the acceleration of the velocity field. The third term accounts for the convection of momentum by the water transported to or from the adjacent storing regions. When $\gamma = 0$, there is no convection. This means that for instance water emerging from the storing regions and joining the current in the streambed, derives its momentum entirely from the motion in the streambed.

The last term of (203), the resistance slope i_r, represents the dissipative forces which are all quadratic in v so that we may put

(204)
$$i_r = \frac{|\,Q\,|\,Q}{K^2} = w\,|\,Q\,|\,Q.$$

Here K is Bakhmeteff's conveyance and w represents the quadratic resistance per unit length, which depends on the water level h. It can usually be treated sufficiently accurately as a frictional resistance by using Chézy's formula, which yields

(205)
$$w = \frac{1}{K^2} = \frac{1}{C^2 A^2 a_s}.$$

The coefficient C is an empirical quantity.

It be observed that no Coriolis or centrifugal forces occur in (203) because they are irrelevant as far as the flow in the length direction only has to be considered. In computing cross currents over shoals between adjacent channels, it may be necessary to account for Coriolis or centrifugal forces.

In case of a storm tide it may be necessary to introduce a term for the forces exerted by the atmosphere, so that we extent (203) as follows:

$$\frac{\delta H}{\delta x} + \frac{1}{g}\frac{\delta v}{\delta t} - (1-\gamma)\, v\, \frac{b-b_s}{gA}\frac{\delta h}{\delta t} + i_r + i_s + i_a = 0. \tag{206}$$

Here i_s, the wind slope, represents the force exerted by the wind on the surface and i_a represents the barometric gradient.

Other forms of the equations. A drawback of the equations (201) and (203) is that there appear four dependant variables, H and Q as well as h and v. Now it is easier to eliminate H and Q than h and v, but unfortunately this is of little use since at the transitions between the sections the quantities H and Q are to be treated as continuous and not h and v in which jumps are to be accounted for. For this reason we shall at least eliminate $\delta v/\delta t$ and $\delta h/\delta t$ by using the relations $H = h + v^2/2g$ and $Q = Av$. Putting $\gamma = 0$ we obtain

$$\frac{\delta Q}{\delta x} + \frac{1}{1 - v^2/v_c^2}\, b\, \frac{\delta H}{\delta t} - \frac{bmv}{1 - v^2/v_c^2}\frac{\delta Q}{\delta t} = 0 \tag{207}$$

$$\frac{\delta H}{\delta x} + \frac{1 + \beta v^2/v_c^2}{1 - v^2/v_c^2}\, m\, \frac{\delta Q}{\delta t} - \frac{bmv}{1 - v^2/v_c^2}\frac{\delta H}{\delta t} + w\,|\,Q\,|\,Q = 0. \tag{208}$$

Here $m = 1/gA$ denotes the *inertance* per unit length (cf [35]) and $v_c = \sqrt{gA/b_s}$ is the critical velocity (cf 5,3). Moreover β is put for $(b - b_s)/b_s$.

When v is negligible with respect to v_c, the third terms in (207) and (208) are small compared to the other terms. Then the following simplification is justified:

$$\frac{\delta Q}{\delta x} + b\frac{\delta H}{\delta t} = 0 \tag{209}$$

$$\frac{\delta H}{\delta x} + m\frac{\delta Q}{\delta t} + w\,|\,Q\,|\,Q = 0. \tag{210}$$

In this case we may as a rule put m constant and this will often be justified likewise with b and w.

If v/v_c is so great that the third terms in (207) and (208) may not be dropped, still in most cases $v^2 \ll v_c^2$, so that we arrive at

$$\frac{\delta Q}{\delta x} + b\frac{\delta H}{\delta t} - 2UbQ\frac{\delta Q}{\delta t} = 0 \tag{211}$$

$$\frac{\delta H}{\delta x} + m\frac{\delta Q}{\delta t} - 2UbQ\frac{\delta H}{\delta t} + w\,|\,Q\,|\,Q = 0, \tag{212}$$

where $U = 1/2gA^2 = \frac{1}{2}gm^2$ is the kinetic factor (UQ^2 is the velocity head). In (211) and (212) we may treat U as a constant as a rule, and b, m and w as functions of H, hence neglecting $v^2/2g$ in the determination of these coefficients.

As the tidal motion is often largely subcritical (cf 5, 3), the equations (211) and (212) are sufficiently correct usually.

Energy equation. In case of designing a tidal power plant the energy equation becomes relevant:

$$(213) \quad \frac{\delta}{\delta x}(\rho g H Q) + \frac{\delta}{\delta t}(\tfrac{1}{2}\rho A v^2) + \rho g b h \frac{\delta h}{\delta t} + (2\gamma - 1)\,\tfrac{1}{2}\rho v^2 (b - b_s)\frac{\delta h}{\delta t} + $$
$$+ \rho g H q_i + \rho g Q i_r = - \rho g Q\,(i_s + i_a).$$

This equation can either be deduced directly from the law of conservation and dissipation of energy, or by adding $\rho g H$ times (202) to $\rho g Q$ times (206).

2, 3. Particularizing conditions.

When the schematization of an estuary has been fixed and the coefficients of Chézy have been determined, a tidal motion in the estuary can be defined by a set of particularizing conditions, usually involving boundary conditions (the number of which depends on the complexity of the estuary system) and two initial conditions in the whole system or the equivalent of them.

Where an estuary debouches in sea, the tidal motion in the sea generates the motion in the estuary. After careful consideration of the interaction of the two bodies of water, it is as a rule possible to set up a boundary condition for the estuary involving the total head at the offing as given function of time.

At the landward end of an estuary we have generally a condition involving the discharge. At a closed end the discharge is obviously zero, and up a tidal river the discharge must approach the fluvial discharge asymptotically.

The computation of the tidal motion in an estuary must moreover observe boundary conditions at every transition between different channels.

When a channel is continued by another channel of different cross-section, it should as a rule be assumed that the total head at the junction is the same in the extremities of both channels. The discharge is likewise the same. These are likewise the conditions to be imposed at the transition between two sections of an estuary where no particular interference with head or discharge prevails. If there is a narrow pass or another obstacle between the two channels, a loss of head must be accounted for, and when water is discharged to or from the junction from aside, e.g. by a sluice, a difference in discharge in the two channels is introduced.

At a junction of three or more channels, the sum of the discharges through the channels to the junction is zero. There are moreover conditions for the differences in head between the channels.

Generally there are in total as many conditions at a junction as there are channels meeting there, and hence there is one boundary condition per extremity of a channel.

When the heads and discharges at a definite instant are given throughout the whole estuary, we can use this as a double initial condition.

Often it is very difficult to obtain sufficient direct observational data to construct such a double initial condition. It is therefore of great practical value that other more suitable conditions equivalent to the initial conditions are possible.

Firstly we may consider a purely periodic tide. Then the condition that all heads and discharges are periodic functions of the time with a given period, replaces the initial conditions.

Secondly we may use the fact that the influence of an initial condition on the subsequent motion decays and dies out gradually. It is therefore possible to compute correctly the tidal motion in an interval of time in which we are interested, by starting from inaccurate initial conditions, provided these conditions lie sufficiently far in the past. The time of decay to be observed depends on the degree of inaccuracy of the initial conditions and on the properties of the estuary system, in particular its extent.

The boundary and initial or periodicity conditions define the particular motion under consideration, which may belong to one of the following types:

1. An ordinary tide, on a particular day.
2. An average tide, usually a lunar mean tide, either diurnal or semi-diurnal.
3. An average spring tide or an average neap tide.
4. A particular observed storm tide.
5. A hypothetical storm tide, generally of excessive height.
6. A tide on a river encountering a fluvial flood.

When there are more observational data on a tidal motion than needed to supply the necessary particularizing conditions, the redundant data may be used for checking. For the Chézy coefficient is so much liable to variations and moreover related so closely to the manner of schematizing, that a check as mentioned is practically indispensable in most cases.

In principle the value of C should be determined for each section separately and as a function of time. Judging from the computations of Faure [34] for the Gironde estuary, the variations in C may then be very considerable. According to the Dutch practice however, variations in C can as a rule be made relatively small by careful schematization, although rather great deviations near slack water cannot always be eliminated. This is of little practical consequence since the resistance near slack water is weak and hence relatively great errors in C are permissable then. For this reason good results are obtained by taking C constant throughout large parts of the estuary system and throughout the entire tidal period or the entire flood or ebb interval. The value of C varies from about 50 $m^{\frac{1}{2}}/sec$ in the shallower rivers to 70 $m^{\frac{1}{2}}/sec$ in the deep inlets.

When a new canal is dug or when an estuary or part of it is modified radically, the value of C has to be assumed. It is then recommendable to estimate the possible deviation of the assumed value and to compute the influence of such a deviation.

3. INTEGRATION BY HARMONIC COMPONENTS

In this chapter we confine ourselves to the periodic tide. First the simplest method to deal with such a tide is expounded: the equations are linearized which makes it possible to consider the tide as sinusoidal (3, 1). Next the nonlinear terms are treated and the interaction of harmonic components is investigated. The formulae dealing with a second harmonic are developed more in detail (3,2—3,4).

3, 1. Single-harmonic method.

Suppose that b, m and w in (209) and (210) may approximately be put constant. Then all terms in these equations are linear, except the resistance term which is nonlinear in Q.

Now consider an estuary without or with little fluvial discharge, where the tidal currents vary approximately by a sinusoidal trend. According to Lorentz [12] we may then replace the quadratic resistance by a linear resistance $r\,Q$ where

$$r = w \frac{8}{3\pi} \mid \mathbf{Q} \mid = 0.85\, w \mid \mathbf{Q} \mid. \tag{301}$$

Here $\mid \mathbf{Q} \mid$ denotes the amplitude of the tidal flow.

The relation (301) was set up by Lorentz on the assumption that the dissipation by the fictitious linear resistance should equal that by the real quadratic resistance. Afterwards Mazure [17] showed that (301) can be obtained as well by a harmonic analysis. This analysis can also be applied if there is an appreciable fluvial discharge Q_0, in combination with a tidal flow $Q_1 = \text{re}\ \mathbf{Q} \exp j\omega t$. Then we find

$$w \mid Q_0 + Q_1 \mid (Q_0 + Q_1) = r_0\, Q_0 + r_1\, Q_1 + \text{higher harm.},$$

where

a) $r_0 = wk_0 \mid \mathbf{Q} \mid \approx w(1.27 \mid \mathbf{Q} \mid + 0.23\, Q_0^2 / \mid \mathbf{Q} \mid)$; b) $r_0 = w\,(Q_0 + \frac{1}{2} \mid \mathbf{Q} \mid {}^2/Q_0)$

and

(302) a) $r_1 = 2wk_1 \mid \mathbf{Q} \mid \approx w\,(0.85 \mid \mathbf{Q} \mid + 1.15\, Q_0^2 / \mid \mathbf{Q} \mid)$; b) $r_1 = 2wQ_0$

when a) $Q_0 < \mid \mathbf{Q} \mid$ or b) $\mid \mathbf{Q} \mid < Q_0$.

The formulae sub a are approximations deduced from (330) (cf 3, 3).

We can separate the mean motion Q_0 and the tide Q_1 (cf 3, 4), and here we shall confine ourselves to the tide. Then we determine a linear resistance r by (301) or (302), using estimations for $\mid \mathbf{Q} \mid$ and if necessary Q_0, to be checked afterwards, and obtain the linearized equations

$$\frac{\delta Q}{\delta x} + b \frac{\delta H}{\delta t} = 0 \tag{303}$$

$$\frac{\delta H}{\delta x} + m \frac{\delta Q}{\delta t} + r\,Q = 0, \tag{304}$$

in which b, m and r are now to be considered as given functions of x. We shall for the moment confine ourselves to the case that b, m and r are constants, at least sectionwise. In an appendix we shall deal briefly with the variability of b, m and r.

The equations (303) and (304) admit periodic solutions of the sinusoidal form

$$H = \mathrm{re}\, \mathbf{H}\, e^{j\omega t} = |\mathbf{H}| \cos(\omega t + \arg \mathbf{H}) \tag{305}$$

$$Q = \mathrm{re}\, \mathbf{Q}\, e^{j\omega t} = |\mathbf{Q}| \cos(\omega t + \arg \mathbf{Q}). \tag{306}$$

Here **H** and **Q**, satisfying the ordinary differential equations

$$\frac{d\mathbf{Q}}{dx} + j\omega b\, \mathbf{H} = 0 \tag{307}$$

$$\frac{d\mathbf{H}}{dx} + (j\omega m + r)\, \mathbf{Q} = 0, \tag{308}$$

denote the complex amplitudes of the vertical and horizontal tide, i.e. the modulus represents the amplitude and the argument represents the phase of the tide.

Both for the physical discussion and for the practical solution of the above equations it is convenient to introduce the *tidal impedance* $\mathbf{Z} = \mathbf{H}/\mathbf{Q}$ and the *tidal admittance* $\mathbf{Y} = 1 / \mathbf{Z} = \mathbf{Q}/\mathbf{H}$ (cf [28] Ch. 4 sect. 23). $|\mathbf{Y}|$ represents the quotient of the amplitudes of horizontal and vertical tide whereas arg **Y** corresponds to the angle of phase lead of the horizontal with respect to the vertical tide. From (307) and (308) it can be deduced that **Z** or **Y** must satisfy the differential equation of Riccati

$$\text{a)}\ \frac{d\mathbf{Z}}{dx} = \mathbf{y}_p\, \mathbf{Z}^2 - \mathbf{z}_s \qquad \text{or b)}\ \frac{d\mathbf{Y}}{dx} = \mathbf{z}_s\, \mathbf{Y}^2 - \mathbf{y}_p, \tag{309}$$

where $\mathbf{y}_p = j\omega b$ and $\mathbf{z}_s = j\omega m + r$.

The general solution of (309a) or b) is

$$\text{a)}\ \mathbf{Z} = \frac{\mathbf{Z}_o - \mathbf{Z}_e \quad \tanh \mathbf{k}\, x}{1 \ - \mathbf{Z}_o\, \mathbf{Y}_e \tanh \mathbf{k}\, x} \quad \text{or b)}\ \mathbf{Y} = \frac{\mathbf{Y}_o - \mathbf{Y}_e \quad \tanh \mathbf{k}\, x}{1 \ - \mathbf{Y}_o\, \mathbf{Z}_e \tanh \mathbf{k}\, x}, \tag{310}$$

where $\mathbf{Z}_o$ or $\mathbf{Y}_o$ is an integration parameter, whereas furthermore

$$\mathbf{k} = \sqrt{\mathbf{y}_p\, \mathbf{z}_s} \quad ; \quad \mathbf{Z}_e = \sqrt{\mathbf{z}_s / \mathbf{y}_p} \quad ; \quad \mathbf{Y}_e = 1 / \mathbf{Z}_e = \sqrt{\mathbf{y}_p / \mathbf{z}_s}.$$

From any solution $\mathbf{Z}(x)$ or $\mathbf{Y}(x)$ we can derive solutions for **Q** and **H** by (307) or (308):

$$\text{a)}\quad \mathbf{Q} = \mathbf{Q}_o \exp -\mathbf{K}_Q(x) \qquad \text{and b)} \quad \mathbf{H} = \mathbf{Q}_o\, \mathbf{Z}(x) \exp - \mathbf{K}_Q(x) \tag{311}$$

or

$$\text{a)}\quad \mathbf{H} = \mathbf{H}_o \exp - \mathbf{K}_H(x) \qquad \text{and b)} \quad \mathbf{Q} = \mathbf{H}_o\, \mathbf{Y}(x) \exp - \mathbf{K}_H(x), \tag{312}$$

where

$$\mathbf{K}_Q = \int_o^x \mathbf{Z}\, \mathbf{y}_p\, dx \qquad \text{or} \qquad \mathbf{K}_H = \int_o^x \mathbf{Y}\, \mathbf{z}_s\, dx,$$

and where $\mathbf{H}_o$ or $\mathbf{Q}_o$ is an integration parameter.

From (310a) and (311) or from (310b) and (312) we deduce the general solution for **H** and **Q** :

$$\mathbf{H} = \mathbf{H}_o \cosh \mathbf{k}\, x - \mathbf{Z}_e\, \mathbf{Q}_o \sinh \mathbf{k}\, x \tag{313}$$

$$\mathbf{Q} = \mathbf{Q}_o \cosh \mathbf{k}\, x - \mathbf{Y}_e\, \mathbf{H}_o \sinh \mathbf{k}\, x. \tag{314}$$

This may also be obtained by more conventional methods from (307) and (308) or by using the particular solutions to the dicussion of which we are now proceeding:

We put in particular $\mathbf{Y}_o = \pm \mathbf{Y}_e$ in (310b). This yields the *elementary solutions* $\mathbf{Y} = \mathbf{Y}_e$ and $\mathbf{Y} = -\mathbf{Y}_e$ which are constant.

From this we derive the solutions

$$\mathbf{H} = \mathbf{H}_o \exp - \mathbf{k}\, x \qquad \text{and} \qquad \mathbf{Q} = \mathbf{H}_o \mathbf{Y}_e \exp - \mathbf{k}\, x,$$

for $\mathbf{H}$ and $\mathbf{Q}$, from which follows

$$H = |\,\mathbf{H}_o\,|\, e^{-(\mathrm{re}\,\mathbf{k})\,x} \cos\, [\omega t - (\mathrm{im}\,\mathbf{k})\, x + \arg \mathbf{H}_o]$$

and

$$Q = |\,\mathbf{H}_o\,|\,|\,\mathbf{Y}_e\,|\, e^{-(\mathrm{re}\,\mathbf{k})\,x} \cos\, [\omega t - (\mathrm{im}\,\mathbf{k})\, x + \arg \mathbf{H}_o + \arg \mathbf{Y}_e].$$

This represents a harmonic wave with the *wave length* (2π / im $\mathbf{k}$) and travelling with the *phase velocity* (ω / im $\mathbf{k}$) in the positive sense of x. The wave is purely periodic in t and damped periodic in x. The damping is exponential at the rate (re $\mathbf{k}$) per unit length. Hence $\mathbf{k}$ is called the complex *propagation exponent* per unit length. The horizontal tide Q leads by the phase angle (arg $\mathbf{Y}_e$) with respect to the vertical tide H.

The solutions

$$\mathbf{H} = \mathbf{H}_o \exp \mathbf{k}\, x \qquad \text{and} \qquad \mathbf{Q} = -\mathbf{H}_o \mathbf{Y}_e \exp \mathbf{k}\, x,$$

derived from $\mathbf{Y} = -\mathbf{Y}_e$, represent waves travelling in the negative sense (cf [28] Ch. 4, sect. 23).

The interference of two waves travelling in opposite senses is represented by superposition of the corresponding solutions. In this way we arrive at

(315) $$\mathbf{H} = \mathbf{H}^+ \exp - \mathbf{k}\, x + \mathbf{H}^- \exp \mathbf{k}\, x$$

(316) $$\mathbf{Q} = \mathbf{Y}_e \mathbf{H}^+ \exp - \mathbf{k}\, x - \mathbf{Y}_e \mathbf{H}^- \exp \mathbf{k}\, x.$$

Here $\mathbf{H}^+$ and $\mathbf{H}^-$ are integration parameters. The reader may verify that (315) and (316) are an alternative form of the general solution (313) and (314), by putting $\mathbf{H}_o = \mathbf{H}^+ + \mathbf{H}^-$ and $\mathbf{Z}_e \mathbf{Q}_o = \mathbf{H}^+ - \mathbf{H}^-$.

When we consider another particular solution $\mathbf{Z}(x)$ or $\mathbf{Y}(x)$, we arrive at other types of solutions for $\mathbf{H}$ and $\mathbf{Q}$. By putting $\mathbf{Y}_o = 0$ in (310b) for instance, and then substituting for $\mathbf{Y}$ in (312), we obtain all the solutions for which $Q = 0$ at $x = 0$. In a similar way $\mathbf{Z}_o = 0$ in (310a) yields all the solutions for which $H = 0$ at $x = 0$. Such solutions may be interpreted as standing harmonic waves (cf [28] Ch. 4, sect. 23).

Appendix to 3, 1.

In order to deal with the variations of b, m and r in dependence on x, we divide the estuary in sections so small that in each of them we are allowed to take mean values for b, m and r. We may then apply (313) and (314) from section to section. This demands much computing labour which often can be reduced considerably by making use of the functions $\mathbf{Y}$ and $\mathbf{Z}$, in particular when we can set up a boundary condition for $\mathbf{Y}$ or $\mathbf{Z}$; this is often possible. Then we compute $\mathbf{Y}$ or $\mathbf{Z}$ by (310b) or (310a) from section to section, and thence deduce $\mathbf{H}$ and $\mathbf{Q}$.

A slightly different procedure was followed by Dronkers [21], who first computed the argument of **Y** by relatively long sections, utilizing the fact that on many rivers arg **Y** varies slowly with x.

In many cases the sections have to be so short in view of the variability of b, m or r, that the integration procedure can be simplified to a finite difference calculus. Suppose there is a boundary condition for **Z**. Then **Z** is computed from section to section by finite differences as follows:

Let $\mathbf{Z}_a$ and $\mathbf{Z}_b$ be the values of **Z** at the ends of a section (x_a, x_b) with the length $l = x_b - x_a$. Then by (310a) approximately

(317) $$\mathbf{Z}_b - \mathbf{Z}_a = \mathbf{Y}_p \mathbf{Z}_m^2 - \mathbf{Z}_s,$$

where $\mathbf{Y}_p = \mathbf{y}_p l = j\omega B$ is the *parallel admittance* of the section and $\mathbf{Z}_s = \mathbf{z}_s l = j\omega M + R$ is its *series impedance*. Moreover $\mathbf{Z}_m = \frac{1}{2}(\mathbf{Z}_a + \mathbf{Z}_b)$. When either $\mathbf{Z}_a$ or $\mathbf{Z}_b$ is known, $\mathbf{Z}_m$ can easily be estimated fairly correctly and then a construction according to (317) yields $\mathbf{Z}_b$ or $\mathbf{Z}_a$ respectively. The estimation of $\mathbf{Z}_m$ is then checked and if necessary the construction is repeated.

For numerical computing it is more convenient to modify (317) into

(318) $$\mathbf{Z}_b - \mathbf{Z}_a = \mathbf{Y}_p \mathbf{Z}_a \mathbf{Z}_b - \mathbf{Z}_s,$$

from which either $\mathbf{Z}_b$ or $\mathbf{Z}_a$ is easily solved when $\mathbf{Z}_a$ or $\mathbf{Z}_b$ is known.

If **Z** becomes too great (say $|\mathbf{Z}| \gg \sqrt{\mathbf{Z}_s / \mathbf{Y}_p}$) the variations $\mathbf{Z}_b - \mathbf{Z}_a$ become excessive and integration of (310b) is more accurate then.

In fig. 2a a graphical construction for the Panama sea level canal is represented. If the Caribbean Sea were entirely tideless, the boundary condition $\mathbf{Z} = 0$ would hold good there. Since there is some tidal motion $\mathbf{H}_A$ (which is given), $\mathbf{Z}_A$ is not zero but relatively small. This small value can be computed with a very satisfactory absolute accuracy by $\mathbf{Z}_A = \mathbf{H}_A / \mathbf{Q}_A$, even if we use a rather crude estimation for $\mathbf{Q}_A$. Such an estimation may be obtained as discussed further below. Hence we start from $\mathbf{Z}_A$ as boundary condition and construct **Z** sectionwise from A to P by (317) and then determine **Q** by (311a),

$$\mathbf{Q} = \mathbf{Q}_P \exp - \mathbf{K}_Q, \text{ where } - \mathbf{K}_Q = \sum_{x_i}^{x_p} \mathbf{Y}_p \mathbf{Z}_m$$

is computed sectionwise; furthermore $\mathbf{Q}_P = \mathbf{H}_P / \mathbf{Z}_P$ follows from the given Pacific vertical tide. Finally $\mathbf{H} = \mathbf{Z}\,\mathbf{Q}$ in virtue of (311b).

Additions or subtractions are performed by vector construction in the diagram whereas the multiplications are performed by adding arguments constructively and multiplying moduli by means of a slide rule.

After having finished the constructions the estimated discharges used in defining the resistance by (301), and moreover $\mathbf{Z}_A$, are checked and the computation is repeated if necessary.

The computation was executed for a schematized canal of 72 km length, 180 m width and a depth below mean level varying from 18 m at the Atlantic to 21 m at the Pacific end. Chézy's coefficient was put 74 $m^{\frac{1}{2}}$/sec. These are the assumptions of

Lamoen [25]. For comparison the harmonic analysis of the results of an exact computation by characteristics (cf 5, 3 and fig. 6) are likewise represented in fig. 2.

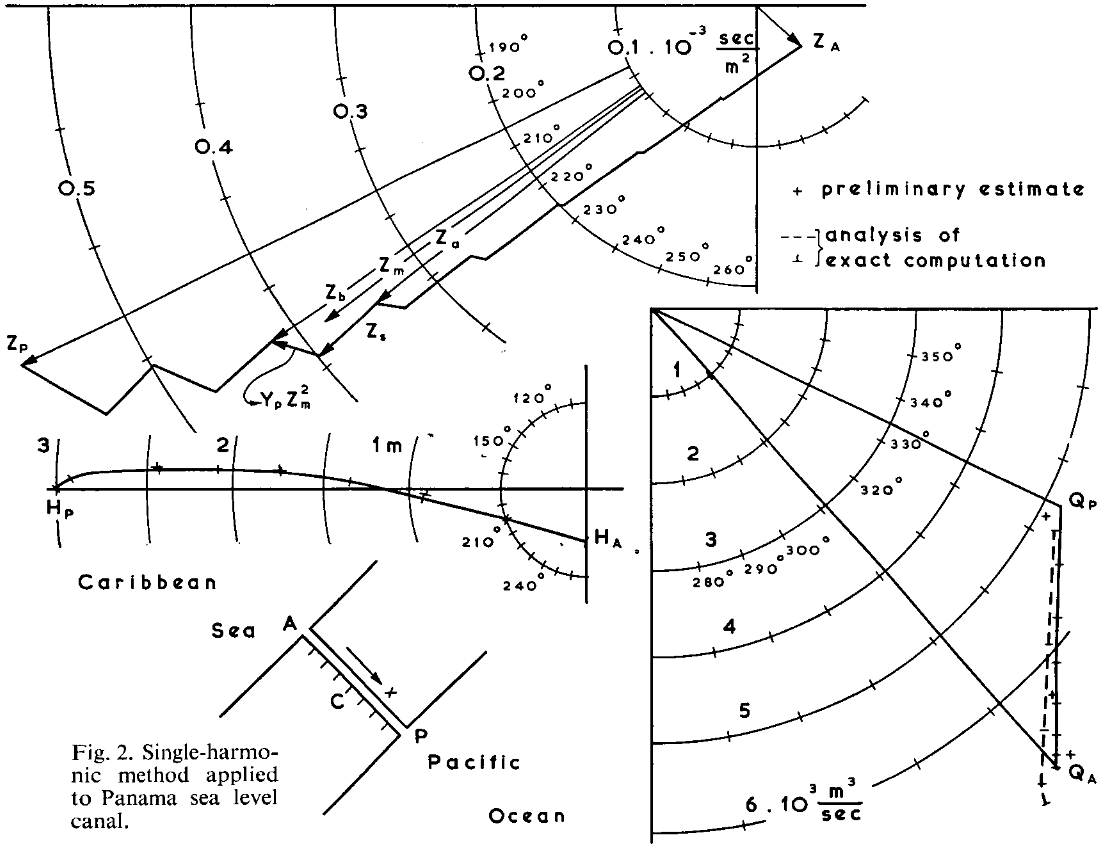

Fig. 2. Single-harmonic method applied to Panama sea level canal.

A simple way to estimate fairly correctly the discharges in the canal, is as follows:

Let W, M and B denote the resistance, inertance and storing area of the whole canal. Let $\mathbf{Q}_C$ represent the discharge in the middle C of the canal. Now according to (301) we put $R = 0.85\ W \mid \mathbf{Q}_C \mid$ and then we deduce from (308) the approximation

$$\mathbf{H}_A - \mathbf{H}_P = (j\omega M + 0.85\ W \mid \mathbf{Q}_C \mid)\, \mathbf{Q}_C\,,$$

where the left-hand member is known. Taking absolute values of both members yields a quadratic equation in $\mid \mathbf{Q}_C \mid^2$ with a unique solution by virtue of $\mid \mathbf{Q}_C \mid$ being real and positive. After substitution of $\mid \mathbf{Q}_C \mid$ in (319), $\mathbf{Q}_C$ can be solved.

Then we put $\mathbf{H}_C = \frac{1}{2}(\mathbf{H}_A + \mathbf{H}_P)$ and compute with the aid of (307):

$$\mathbf{Q}_A = \mathbf{Q}_C + \tfrac{1}{2} j\omega B (\tfrac{1}{2}\mathbf{H}_A + \tfrac{1}{2}\mathbf{H}_C) \quad ; \quad \mathbf{Q}_P = \mathbf{Q}_C - \tfrac{1}{2} j\omega B (\tfrac{1}{2}\mathbf{H}_P + \tfrac{1}{2}\mathbf{H}_C).$$

These results for $\mathbf{Q}$, which are represented in fig. 2, can be used as basis for the above more detailed analysis.

3, 2. Preparations for multiple harmonic methods.

The approximation of a tide by a simple sine function, however useful for exploring a tidal problem roughly, is too crude in many cases when a more detailed investigation of the tidal phenomena is demanded. We can then try to treat the tidal motion as a purely periodic phenomenon composed of a fundamental and higher harmonic components. The period will as a rule be the period of the lunar tide (12 hours 25 minutes).

The computation of the fundamental component is relatively easy as long as the higher harmonic components are not too strong (say less than 40% of the fundamental). The influence of the latter on the fundamental is negligible then, so that the fundamental may be computed substantially along the lines of the preceding section.

The higher harmonic components demand much more computation labour and this labour increases disproportionally with the number of harmonic components to be computed, owing to the strong mutual interaction of higher harmonic components. This is associated with the fact that the deviations of a tidal curve from the simple sine form, or from a combination of a zero, a fundamental and a second harmonic component, are generally not well represented by one single higher harmonic component.

In regularly shaped rivers the second harmonic component is as a rule a fraction of the fundamental, and the third is a fraction of the second harmonic (in the Dutch rivers a half or less, and one third respectively). In such circumstances a computation of zero, first and second harmonic component only, will meet most practical requirements. Occasionally the higher harmonics are so small that they may be neglected altogether.

In other cases, e.g. in more irregularly shaped rivers and estuaries, the second and third harmonic are possibly appreciable and of equal order of magnitude. It would then be necessary to compute the third harmonic as well since the first and third harmonics together produce a second and other harmonics owing to the non-linear terms in the differential equations, in particular the quadratic resistance.

It may be assumed that it is economic as a rule to compute the second harmonic. When this is no longer sufficient so that further components are required, abandoning the harmonic method for an exact method, e.g. by a characteristic analysis (cf Ch. 5), is usually preferable. For that reason we shall hereafter confine ourselves to developing the formulae for the zero, first and second harmonic. The formulae for the third and higher harmonics may be derived if necessary along similar lines of thought (cf the appendix to 3,3).

We base ourselves on (211) and (212) where we treat b, m and w as functions of H ; we put U constant.

We consider a periodic tide with period Θ and fundamental angular frequency $\omega = 2\pi / \Theta$ and expand H and Q in Fourier series. Hence

$$H = H_0 + H_1 + H_2 \ldots ,$$

where

$$H_n = \mathbf{H}_n e^{jn\omega t} + \overline{\mathbf{H}}_n e^{-jn\omega t} = 2 \mid \mathbf{H}_n \mid \cos (n\omega t + \arg \mathbf{H}_n).$$

Here $\overline{\mathbf{H}}_n$ denotes the complex conjugate of $\mathbf{H}_n$. The constant H_0 is the mean head, H_1 the fundamental tide, H_2 the second harmonic tide, etc. The modulus of the complex constant $\mathbf{H}_n$ represents half the amplitude of the n-th harmonic component and arg $\mathbf{H}_n$ its phase. So $\mathbf{H}$ of the preceding section is $2\mathbf{H}_1$.

In the same way we analyse Q:

$$Q = Q_0 + Q_1 + Q_2 + \ldots . ; \; Q_n = \mathbf{Q}_n e^{jn\omega t} + \mathrm{cc}\,(n \geqslant 1).$$

Here Q_0 is the mean discharge (on a river identical with the fluvial discharge). Furthermore cc denotes the complex conjugate of the preceding term.

Usually higher harmonic components are weaker than lower ones. Yet this is not at all a rule without exceptions. We shall assume however, that the fundamental dominates over the second and higher harmonic components. Then we may usually assume moreover that the variations of b, m and w in the course of time are substantially defined by the fundamental vertical tide. So we put

(319) $$b = b^{(0)} + b^{(1)} H_1 .$$

Here

$$b^{(0)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} b\,(H_0 + 2 \mid \mathbf{H}_1 \mid \cos \vartheta)\, d\,\vartheta\;;$$

$$b^{(1)} \mid \mathbf{H}_1 \mid = \frac{1}{2\pi} \int_{-\pi}^{\pi} b\,(H_0 + 2 \mid \mathbf{H}_1 \mid \cos \vartheta) \cos \vartheta \, d\,\vartheta\,,$$

where

$$\vartheta = \omega t + \arg \mathbf{H}_1 .$$

Usually these coefficients $b^{(0)}$ and $b^{(1)}$ are almost independent of the amplitude $2 \mid \mathbf{H}_1 \mid$.

Similarly we put

(320) $$m = m^{(0)} - m^{(1)} H_1 ,$$

where $m^{(0)}$ and $m^{(1)}$ may be defined in the same way as $b^{(0)}$ and $b^{(1)}$. Instead we may apply an analysis as following below for w.

In a channel with a rectangular cross section we have

$$w = \frac{1}{C^2\, b_s^2\, a^3}.$$

Now neglecting the velocity head we may put

$$a = a_0 + H_1 + H_2 ,$$

where a_0 is the average depth during the tidal period. As H_2 is assumed to be small compared to $a_0 + H_1$, and $H_1 < a_0$, we expand in powers of H_2 as follows

$$w = \frac{1}{C^2 b_s^2 (a_0 + H_1)^3} - \frac{3}{C^2 b_s^2 (a_0 + H_1)^4} H_2 + \ldots\ldots$$

For the sake of brevity we shall further omit the terms with H_2 which are often negligible. Then we can put

(321) $$w = w^{(0)} - w^{(1)} (\mathbf{H}_1 e^{j\omega t} + \text{cc}) + w^{(2)} (\mathbf{H}_1^2 e^{2j\omega t} + \text{cc}),$$

where the coefficients $w^{(0)}$, $w^{(1)}$ and $w^{(2)}$ are defined by Fourier analysis of the factor

$$\frac{1}{(a_0 + H_1)^3} = \frac{1}{[a_0 + 2 \mid \mathbf{H}_1 \mid \cos (\omega t + \arg \mathbf{H}_1)]^3}.$$

This yields

$$w^{(0)} = \frac{c_0^{(3)}}{C^2 b_s^2 a_0^3} = \frac{a_0^2 + 2 \mid \mathbf{H}_1 \mid^2}{C^2 b_s^2 \sqrt{a_0^2 - 4 \mid \mathbf{H}_1 \mid^2}^{\,5}}$$

$$\mid \mathbf{H}_1 \mid . \, w^{(1)} = \frac{c_1^{(3)}}{C^2 b_s^2 a_0^3} = \frac{3a_0}{C^2 b_s^2 \sqrt{a_0^2 - 4 \mid \mathbf{H}_1 \mid^2}^{\,5}} . \mid \mathbf{H}_1 \mid$$

$$\mid \mathbf{H}_1 \mid^2 . \, w^{(2)} = \frac{c_2^{(3)}}{C^2 b_s^2 a_0^3} = \frac{6}{C^2 b_s^2 \sqrt{a_0^2 - 4 \mid \mathbf{H}_1 \mid^2}^{\,5}} . \mid \mathbf{H}_1 \mid^2.$$

For the general definition of the coefficients $c_n^{(\gamma)}$ cf [28] Ch 14 sec 11.

It now remains to analyse the quadratic factor $\mid Q \mid Q$ in the resistance. In view of the importance of this factor we shall devote a separate section to it.

3, 3. Analysis of the quadratic resistance.

In case Q keeps the same sign, say $+$, throughout the entire period, the analysis of the quadratic factor $\mid Q \mid Q = Q^2$ offers no particular difficulties. Neglecting third and higher harmonics in Q as well as in Q^2, we find by simply executing the multiplication of the series for Q with itself:

(322) $$Q^2 = (Q_0^2 + 2 \mid \mathbf{Q}_1 \mid^2 + 2 \mid \mathbf{Q}_2 \mid^2) + [(2 Q_0 \mathbf{Q}_1 + 2 \overline{\mathbf{Q}}_1 \mathbf{Q}_2) e^{j\omega t} + \text{cc}] + \\ + [(\mathbf{Q}_1^2 + 2Q_0 \mathbf{Q}_2) e^{2j\omega t} + \text{cc}].$$

When Q changes sign during the period, the analysis of $\mid Q \mid Q$ becomes much more complicated. The first who treated this problem was Mazure [17]. He confined himself to the case that Q is a simple sine function,

$$Q = Q_0 + (\mathbf{Q}_1 e^{j\omega t} + \text{cc}) = Q_0 + 2 \mid \mathbf{Q}_1 \mid \cos (\omega t + \arg \mathbf{Q}_1).$$

Then $S = \mid Q \mid Q$ is a non-sinusoidal periodic function which can be decomposed in a mean value S_0, a fundamental S_1 etc. Mazure demonstrated that, in case $Q_0 = 0$, this fundamental S_1 is exactly Lorentz' linearized resistance defined by imposing the condition that the linearization should yield the true dissipation of energy during an entire period.

After the work of Mazure it has been tried to extend the theory by considering also the higher harmonic components. This encounters great practical difficulties however as we explain below:

In order to perform the integrals of the Fourier analysis the instants at which the flow turns have to be determined, because at those instants the factor $S = |Q|Q = \pm Q^2$ changes sign likewise. The instants of slack water are defined by goniometric equations. Even if we neglect the third and higher harmonics in Q, this goniometric equation is still equivalent with a quartic algebraical equation and therefore it is not possible to represent the roots by a simple formula. Consequently the results of the Fourier analysis can not be brought into a workable form either.

Therefore we must have recourse to approximate procedures. One of these procedures consists in approximating the instants of slack water by the zeros of $Q_0 + Q_1$. This we treat below in connection with Schönfeld's turning function (cf [28] Ch 14, sect 122). In an appendix we shall deal with more refined approximations.

We introduce a turning function T defined by

$$T(t) = \begin{cases} +1 & \text{if} \quad Q > 0 \\ -1 & \text{if} \quad Q < 0, \end{cases}$$

so that we may put $S = |Q|Q = TQ^2$. The Fourier coefficients of T, defined by

$$\mathbf{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(t)\, e^{-jn\omega t}\, d(\omega t),$$

might easily be computed if we knew the instants of slack water. When there are two such instants in a period, we have

$$(323) \quad \begin{aligned} &\text{a)} \quad T_0 = \frac{\omega t_1' - \omega t_1}{\pi} \pm 1 \qquad \begin{array}{l} + \text{ if } t_1 > t_1' \\ - \text{ if } t_1 < t_1' \end{array} \\ &\text{b)} \quad \mathbf{T}_n = \frac{j}{\pi n}\,[e^{-jn\omega t_1'} - e^{-jn\omega t_1}], \end{aligned}$$

where t_1 is the instant at which Q turns to the positive and t_1' the instant at which Q turns to the negative.

Now we shall approximate t_1 and t_1' by the zeros of the function $Q_0 + Q_1$. We assume $Q_0 < 2|\mathbf{Q}_1|$ for otherwise there is usually no slack water at all.

We suppose $Q_0 > 0$ and introduce an auxiliary angle γ by

$$(324) \qquad \cos\gamma = Q_0 / 2|\mathbf{Q}_1|,$$

so that

$$(325) \qquad Q = Q_0 + Q_1 = 2|\mathbf{Q}_1|\,[\cos\gamma + \cos(\omega t + \arg \mathbf{Q}_1)].$$

Hence

$$(326) \qquad \text{a)} \quad \omega t_1 = \pi + \gamma - \arg \mathbf{Q}_1; \qquad \text{b)} \quad \omega t_1' = \pi - \gamma - \arg \mathbf{Q}_1.$$

By substitution of (326) in (323) we deduce

(327) $$T = k_0'' + \sum_{n=1}^{\infty} k_n'' \left(e^{jn \arg \mathbf{Q}_1} \,.\, e^{j\omega t} + \text{cc}\right),$$

where

(328) a) $k_0'' = 1 - \dfrac{2\gamma}{\pi}$ b) $k_n'' = (-1)^{n+1} \dfrac{2 \sin n\gamma}{\pi n}$ $(n \neq 0)$.

Now we proceed further as follows:

(329) $$TQ^2 = T(Q_0 + Q_1 + Q_2 + \ldots.)^2 = T(Q_0 + Q_1)^2 + 2T(Q_0 + Q_1).$$
$$.\,(Q_2 + Q_3 + \ldots.) + T(Q_2 + Q_3 + \ldots.)^2.$$

We confine ourselves, as said before, to the case that Q_3 etc. may be be neglected. So we drop the terms $2\,T(Q_0 + Q_1)\,Q_3$ etc. Then the term $T(Q_2 + \ldots.)^2$ is likewise negligible as a rule.

Applying (326) and (327) and dropping third and higher harmonics yields

(330) $$T(Q_0 + Q_1)^2 = 4\,k_0\,|\,\mathbf{Q}_1\,|^2 + 4\,k_1\,|\,\mathbf{Q}_1\,|\,(\mathbf{Q}_1\,e^{j\omega t} + \text{cc}) +$$
$$+ 4\,k_2(\mathbf{Q}_1^2\,e^{2j\omega t} + \text{cc}),$$

where
$$k_0 = \left(1 - \frac{2\gamma}{\pi}\right)\left(\tfrac{1}{2} + \cos^2\gamma\right) + \frac{3}{2\pi}\sin 2\gamma$$
$$k_1 = \left(1 - \frac{2\gamma}{\pi}\right)\cos\gamma + \frac{1}{\pi}\left(\tfrac{3}{2}\sin\gamma + \tfrac{1}{6}\sin 3\gamma\right)$$
$$k_2 = \tfrac{1}{4}\left(1 - \frac{2\gamma}{\pi}\right) + \frac{1}{\pi}\left(\tfrac{1}{3}\sin 2\gamma - \tfrac{1}{24}\sin 4\gamma\right).$$

The above result conforms to Mazure's analysis.

The introduction of the second harmonic of Q produces a number of terms of which the following are the most important:

(331) $$2T\ (Q_0 + Q_1)\,Q_2 = \frac{4k_2'}{|\,\mathbf{Q}_1\,|}\left(\mathbf{Q}_1^2\,\overline{\mathbf{Q}}_2 + \text{cc}\right) +$$
$$+ 4\,k_1'\left(\overline{\mathbf{Q}}_1\,\mathbf{Q}_2\,e^{j\omega t} + \text{cc}\right) + 4\,k_0'\,|\,\mathbf{Q}_1\,|\left(\mathbf{Q}_2\,e^{2j\omega t} + \text{cc}\right) + \ldots\ldots$$

Here
$$k_0' = \left(1 - \frac{2\gamma}{\pi}\right)\cos\gamma + \frac{2}{\pi}\sin\gamma$$
$$k_1' = \tfrac{1}{2}\left(1 - \frac{2\gamma}{\pi}\right) + \frac{1}{2\pi}\sin 2\gamma$$
$$k_2' = \frac{1}{\pi}\left(\tfrac{1}{2}\sin\gamma - \tfrac{1}{6}\sin 3\gamma\right).$$

For the derivation of these coefficients we must apply (328) for $n = 0$, 1, 2, and 3, and (324) (cf 28 Ch. 14, sec 122).

The above analysis yields fairly accurate results even if the second harmonic component is appreciable, say 40 or 50% of the mean and the fundamental. This is explained as follows:

Dropping the second harmonic only affects the approximations for the instants of slack water. This means that in the interval between the assumed and the real instant of slack water, a wrong sign is appended to Q^2. This is of relatively little consequence however, since Q^2 is small near slack water.

Appendix to 3, 3.

When the higher harmonics in Q are strong, the above analysis is no longer applicable. In this appendix we treat briefly two methods to be considered then.

1. We approximate $|Q|Q$ by a polynomial, e.g. a cubic, as follows:

Let $Q_m + Q_d$ be the greatest and $Q_m - Q_d$ the smallest value of Q during a period in a definite place ($Q_d > Q_m$; otherwise there is no slack water). We introduce the parameter $p = Q_m / Q_d$ $(0 \leqslant p < 1)$ and put $x = (Q - Q_m) / Q_d$. Then we have

$$|Q|Q = Q_d^2 |p + x|(p + x).$$

Now expand $|Q|Q$ in the interval $-1 \leqslant x \leqslant 1$ by the series:

$$|Q|Q = \sum_{n=0}^{\infty} S_n P_n(x),$$

where $P_n(x)$ denotes the polynomials of Legendre. By virtue of the fact that these polynomials are normal, we have

$$\frac{2}{2n+1} S_n = Q_d^2 \int_{-1}^{+1} |p + x| (p + x) P_n(x)\, dx.$$

By these integrals the coefficients S_n are defined as functions of p. We terminate after S_3 and then obtain the cubic approximation:

$$S = |Q|Q \approx n_0 Q_d^2 + n_1 Q_d Q + n_2 Q^2 + n_3 \frac{Q^3}{Q_d}. \tag{332}$$

The coefficients n_0, n_1, n_2 and n_3 are functions of p (cf [28] fig. 105).

By substituting the Fourier series for Q in (332), the series for S is easily deduced.

An alternative approximation in the form of an odd power polynomial of the seventh degree, as deduced by Stroband [20], holds good for the circumstances on the Dutch rivers, but the cubic (332) has a considerably wider range of application.

It has appeared that the procedure by Legendre polynomials is not quite free from objections, which make an extension beyond the third degree not advisible. For this reason recently the problem has been approached from a new angle:

2. We treat the factor $|Q|Q$ by first analyzing $|Q|$ as follows:

From the Fourier series for Q we can easily deduce a series for Q^2. Then we have

$$| Q | = \sqrt{Q^2} = \sqrt{P\,[1 + \varphi(t)]}\,,$$

where

$$P = Q_0^2 + 2 \sum_{p=1}^{n} | \mathbf{Q}_p |^2 \text{ and } \varphi(t) = \sum_{q=1}^{2n} (B_q \, e^{iq\omega t} + \text{cc}),$$

where B_q denotes a set of coefficients depending on the Fourier coefficients of Q.

If max $\varphi(t)$ during a period is less than 1, we can apply the binomial series

$$\sqrt{Q^2} = \sqrt{P} \sum_{p=0}^{\infty} \binom{\frac{1}{2}}{p} \left[\sum_{q=1}^{2n} B_q \, e^{jq\omega t} + \text{cc} \right]^p .$$

In practical applications however, max φ may very well be nearly 1 or greater. In that case we write

$$\sqrt{1 + \varphi(t)} \approx 1 + \tfrac{1}{2}\varphi(t) - a\,\varphi^2(t), \tag{333}$$

where a is defined as follows: let A be an estimate of max φ, then we require a to satisfy

$$\sqrt{1 + A} = 1 + \tfrac{1}{2} A - a\,A^2.$$

The value of a is not very sensitive to variations of A. Now by virtue of

$$\varphi = \frac{Q^2}{P} - 1$$

we have

$$S = | Q | Q \approx Q\sqrt{P}\,[(\tfrac{1}{2} - a) + (\tfrac{1}{2} + 2a)\frac{Q^2}{P} - a\,\frac{Q^4}{P^2}]\,, \tag{334}$$

by which the Fourier coefficients of S can be deduced.

The clue of the above method lies in the fact that (333) is most accurate for the greater values of $| Q |$. Perhaps it is less accurate for small values, but this is of little consequence since then the product $| Q | Q$ is small.

In order to demonstrate the value of the above approximations we consider an example in which the second harmonic is twice as strong as the fundamental:

$$Q = \cos \omega t + 2 \cos 2\omega t.$$

Exact numerical analysis yields

$|Q | Q = 0.36 + 2.62 \cos \omega t + 4. \cos 2\omega t + 0.7 \cos 3\omega t + 0.1 \cos 4\omega t.$

Furthermore we obtain by (332):

$| Q | Q = 0.42 + 2.72 \cos \omega t + 4.17 \cos 2\omega t + 0.9 \cos 3\omega t + 0.4 \cos 4\omega t.$

Finally (334) yields

$| Q | Q = 0.31 + 2.55 \cos \omega t + 4.06 \cos 2\omega t + 0.75 \cos 3\omega t + 0.22 \cos 4\omega t.$

Apparently the latter is the closest approximation.

3, 4. Separation of harmonic components.

The Fourier expressions derived above are now substituted in the terms of the differential equations (211) and (212). Then we have, confining ourselves to zero, first and second harmonics:

$$\frac{\delta Q}{\delta x} = \frac{dQ_0}{dx} + \sum_{n=1}^{2} \left(\frac{d\mathbf{Q}_n}{dx} e^{jn\omega t} + \mathrm{cc} \right).$$

Furthermore (cf (319)):

$$b\frac{\delta H}{\delta t} = b^{(0)} \sum_{n=1}^{2} \frac{\delta H_n}{\delta t} + b^{(1)} H_1 \frac{\delta H_1}{\delta t} =$$

$$= (j\omega b^{(0)} \mathbf{H}_1 e^{j\omega t} + \mathrm{cc}) + (2j\omega b^{(0)} \mathbf{H}_2 e^{2j\omega t} + \mathrm{cc}) + (j\omega b^{(1)} \mathbf{H}_1^2 e^{2j\omega t} + \mathrm{cc}).$$

Here terms which are usually negligibly small have been omitted. In the third term of (211) which is small, we confine ourselves to the terms:

$$-2\,U b\, Q \frac{\delta Q}{\delta t} = -2\,U b^{(0)} (Q_0 + Q_1) \frac{\delta Q_1}{\delta t} =$$

$$= -(2\,j\omega U\, b^{(0)}\, Q_0\, \mathbf{Q}_1\, e^{j\omega t} + \mathrm{cc}) - (2\,j\omega U\, b^{(0)}\, \mathbf{Q}_1^2\, e^{2j\omega t} + \mathrm{cc}).$$

In a similar way we get in the dynamic equation (cf (320), (321), (330) and (331)):

$$\frac{\delta H}{\delta x} = \frac{dH_0}{dx} + \sum_{n=1}^{2} \left(\frac{d\mathbf{H}_n}{dx} e^{jn\omega t} + \mathrm{cc} \right)$$

$$m\frac{\delta Q}{\delta t} = -2\omega m^{(1)} \,|\, \mathbf{H}_1 \,|\,.\,|\, \mathbf{Q}_1 \,|\, \sin \varnothing_1 + (j\omega m^{(0)}\, \mathbf{Q}_1\, e^{j\omega t} + \mathrm{cc}) +$$

$$+ (2j\omega m^{(0)}\, \mathbf{Q}_2\, e^{2j\omega t} + \mathrm{cc}) - (j\omega m^{(1)}\, \mathbf{H}_1\, \mathbf{Q}_1\, e^{2j\omega t} + \mathrm{cc})$$

$$2\,U b\, Q \frac{\delta H}{\delta t} = 4\,\omega\, U\, b^{(0)} \,|\, \mathbf{Q}_1 \,|\,.\,|\, \mathbf{H}_1 \,|\, \sin \varnothing_1 - (2\,j\,\omega\, U\, b^{(0)}\, Q_0\, \mathbf{H}_1\, e^{j\omega t} + \mathrm{cc}) +$$

$$- (2\,j\,\omega\, U\, b^{(0)}\, \mathbf{Q}_1\,.\,\mathbf{H}_1\, e^{2j\omega t} + \mathrm{cc})$$

$$w\,|\,Q\,|\,Q = 4k_0 w^{(0)} \,|\, \mathbf{Q}_1 \,|^2 - 8\,k_1\, w^{(1)} \,|\, \mathbf{H}_1 \,|\,.\,|\, \mathbf{Q}_1 |^2 \cos \varnothing_1 + 8\,k_2'\, w^{(0)} \frac{\mathrm{re}(\mathbf{Q}_1^2\, \overline{\mathbf{Q}}_2)}{|\, \mathbf{Q}_1 \,|} +$$

$$- 8\,k_1'\, w^{(1)}\, \mathrm{re}\,(\mathbf{H}_1\, \mathbf{Q}_1\, \overline{\mathbf{Q}}_2) +$$

$$+ [4\,k_1 w^{(0)} \,|\, \mathbf{Q}_1 \,|\, \mathbf{Q}_1 - 4\,k_0 w^{(1)}\, \mathbf{H}_1 \,|\, \mathbf{Q}_1 |^2 + 4k_1' w^{(0)}\, \overline{\mathbf{Q}}_1 \mathbf{Q}_2 + 4k_1 w^{(2)} \mathbf{H}_1^2 |\mathbf{Q}_1| \overline{\mathbf{Q}}_1$$

$$- 4\,k_2 w^{(1)}\, \overline{\mathbf{H}}_1\, \mathbf{Q}_1^2 - 4\,k_0' w^{(1)}\, \overline{\mathbf{H}}_1 \,|\, \mathbf{Q}_1 \,|\, \mathbf{Q}_2]\, e^{j\omega t} + \mathrm{cc} +$$

$$+ [-4k_1 w^{(1)}\, \mathbf{H}_1 \,|\, \mathbf{Q}_1 \,|\, \mathbf{Q}_1 + 4k_2 w^{(0)}\, \mathbf{Q}_1^2 + 4k_0' w^{(0)} |\mathbf{Q}_1| \mathbf{Q}_2 + 4k_0 w^{(2)} \mathbf{H}_1^2 |\mathbf{Q}_1|^2 +$$

$$- 4\,k_1' w^{(1)}\, \mathbf{H}_1\, \overline{\mathbf{Q}}_1\, \mathbf{Q}_2]\, e^{2j\omega t} + \mathrm{cc}.$$

Here $\varnothing_1 = \pi + \arg \mathbf{Q}_1 - \arg \mathbf{H}_1$ denotes the angle of phase lead of the current fundamental Q_1 with respect to the head fundamental H_1.

When the above expressions have been substituted for the terms of the differential equations (211) and (212), these equations can be resolved into separate equations for each harmonic component.

The terms independent of t must satisfy the equations

(335) $$\frac{dQ_0}{dx} = 0$$

(336) $$\frac{dH_0}{dx} + 4\,k_0\,w^{(0)}\,|\,\mathbf{Q}_1\,|^2 + \underline{\omega\,(4\,U\,b^{(0)} - 2m^{(1)})\,|\,\mathbf{H}_1\,|\,.\,|\,\mathbf{Q}_1\,|\sin \varnothing_1} +$$
$$-8k_1w^{(1)}\,|\,\mathbf{H}_1|\,.\,|\,\mathbf{Q}_1\,|^2\cos\varnothing_1 +$$
$$+\,8k_2'w^{(0)}\,\frac{\mathrm{re}\,(\mathbf{Q}_1^2\,\overline{\mathbf{Q}}_2)}{|\,\mathbf{Q}_1\,|} - 8k_1'w^{(1)}\mathrm{re}\,(\mathbf{H}_1\,\mathbf{Q}_1\,\overline{\mathbf{Q}}_2) = 0.$$

The underlined terms are as a rule small compared to the main terms which are not underlined. Double underlining denotes the smallest terms.

The coefficients of the factor $e^{\,j\omega t}$ must satisfy

(337) $$\frac{d\mathbf{Q}_1}{dx} + j\omega b^{(0)}\,\mathbf{H}_1 - \underline{2\,j\omega\,U\,b^{(0)}\,Q_0\,\mathbf{Q}_1} = 0$$

(338) $$\frac{d\mathbf{H}_1}{dx} + j\omega m^{(0)}\mathbf{Q}_1 - \underline{2j\omega Ub^{(0)}Q_0\mathbf{H}_1} + 4k_1w^{(0)}\,|\mathbf{Q}_1\,|\,\mathbf{Q}_1 - \underline{4k_0w^{(1)}\,\mathbf{H}_1\,|\,\mathbf{Q}_1\,|^2} +$$
$$\underline{+\,4k_1'w^{(0)}\,\overline{\mathbf{Q}}_1\,\mathbf{Q}_2 + 4k_1\,w^{(2)}\,\mathbf{H}_1^2\,|\,\mathbf{Q}_1\,|\,\overline{\mathbf{Q}}_1 - 4k_2\,w^{(1)}\,\overline{\mathbf{H}}_1\mathbf{Q}_1^2 - 4k_0'\,w^{(1)}\,\overline{\mathbf{H}}_1\,|\,\mathbf{Q}_1\,|\,\mathbf{Q}_2} = 0.$$

The underlinings again denote orders of magnitude.

The coefficients of $e^{2j\omega t}$ must satisfy

(339) $$\frac{d\mathbf{Q}_2}{dx} + \underline{2\,j\omega b^{(0)}\,\mathbf{H}_2} + \underline{j\omega b^{(1)}\,\mathbf{H}_1^2} - \underline{2\,j\omega b^{(0)}\,U\,\mathbf{Q}_1^2} = 0$$

(340) $$\frac{d\mathbf{H}_2}{dx} + \underline{2\,j\omega m^{(0)}\,\mathbf{Q}_2} - \underline{j\omega\,(m^{(1)} + 2\,Ub^{(0)})\,\mathbf{H}_1\,\mathbf{Q}_1} - \underline{4\,k_1w^{(1)}\,\mathbf{H}_1\,|\,\mathbf{Q}_1\,|\,\mathbf{Q}_1} +$$
$$+\,\underline{4\,k_2\,w^{(0)}\,\mathbf{Q}_1^2} + \underline{4\,k_0'\,w^{(0)}\,|\,\mathbf{Q}_1\,|\,\mathbf{Q}_2} + \underline{4\,k_0w^{(2)}\,\mathbf{H}_1^2\,|\,\mathbf{Q}_1\,|^2} - \underline{4\,k_1'\,w^{(1)}\,\mathbf{H}_1\,\overline{\mathbf{Q}}_1\,\mathbf{Q}_2} = 0.$$

All these terms are small compared to the main terms in (337) and (338).

The solution is searched for along the following line:
First we neglect the double underlined terms in (337) and (338) and solve the fundamental tide substantially as described in 3, 1.

Then we drop the double underlined terms in (336) and compute H_0 by numerical integration. Here we can be supposed to know Q_0 and one boundary condition for H (estuary or maritime river), or we have two boundary conditions for H (canal between two seas or the like).

Next we substitute the results for the zero and first harmonics in (339) and (340). These equations are linear in $\mathbf{H}_2$ and $\mathbf{Q}_2$ and nonhomogenous. They are solved by applying the theorem that every solution can be expressed as the sum of an arbitrary particular solution and a complementary function being a solution of the homogenized subsidiary equations. An arbitrary particular solution is easily constructed by integration by finite differences from section to section, and complementary functions can be determined substantially as described in 3, 1.

Finally we correct the fundamental and zero harmonics for the double underlined terms.

We conclude by making two remarks:

The influence of the small terms presents an intricate question. In order to justify the neglect of certain terms, it is not sufficient to verify that each of these terms is small. It may be that a rather great number of small terms all have the same sign so that they accumulate. If this occurs, it may be worth while to compute these terms, at least some of them, in order to get an idea of the tendency of their influence.

If other terms than those presented above have to be introduced, e.g. by making use of one of the analyses of the appendix to 3,3, the derivation of the formulae follows substantially the same line.

4. DIRECT INTEGRATION

First the finite difference methods are discussed, both in the original quad-scheme and in the more recent cross-scheme (4, 1). Then the principles of the more refined methods of power-series and iteration are expounded (4,2). This is followed by various applications of these methods (4,3—4,5).

4, 1. Finite difference methods.

Among the first who proposed the numerical computation of tidal movements are de Vries Broekman [7] and Reineke [9]. Both put forward a direct integration of the differential equations by finite differences. More recently Holsters [19] has introduced another method which we consider as a modified application of the idea of finite difference integration.

The efficiency of this kind of methods is greatly improved by arranging the first order differences symmetrically such that the second order differences cancel. In order to discuss this, let the time be divided into relatively small intervals of equal duration $\triangle t$ whereas the estuary is divided into rather short sections. Then a grid is formed in the tx-diagram as illustrated by fig. 3, in which AF may represent an estuary with the inlet at F and its landward extremity at A.

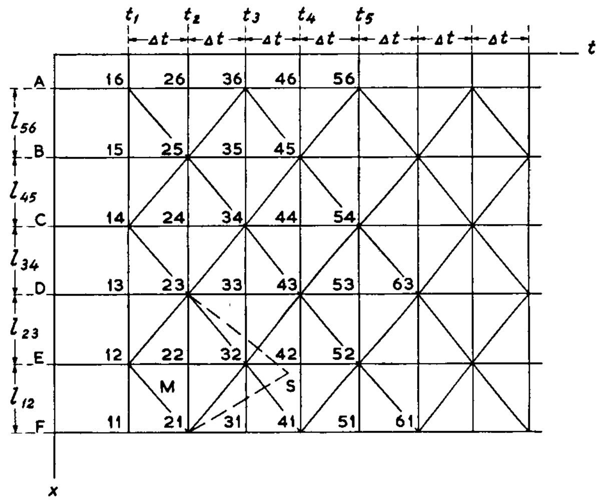

Fig. 3. Grid for difference methods.

Two main procedures are to be distinguished: working by quad-differences or by cross- differences.

Quad-differences. Let Q be expanded in a Taylor series in the environment of the centre M of the rectangle 11-21-22-12. Then

$$Q_{11}=Q_M+\tfrac{1}{2}Q_x l_{12}-\tfrac{1}{2}Q_t\triangle t+\tfrac{1}{8}Q_{xx}l_{12}^2-\tfrac{1}{4}Q_{xt}l_{12}\triangle t+\tfrac{1}{8}Q_{tt}\triangle t^2+\tfrac{1}{48}Q_{xxx}l_{12}^3-$$
$$-\tfrac{1}{16}Q_{xxt}l_{12}^2\triangle t\ldots\ldots$$
$$Q_{21}=Q_M+\tfrac{1}{2}Q_x l_{12}+\tfrac{1}{2}Q_t\triangle t+\tfrac{1}{8}Q_{xx}l_{12}^2+\tfrac{1}{4}Q_{xt}l_{12}\triangle t+\tfrac{1}{8}Q_{tt}\triangle t^2+\ldots\ldots\ldots$$
$$Q_{12}=Q_M-\tfrac{1}{2}Q_x l_{12}-\tfrac{1}{2}Q_t\triangle t+\tfrac{1}{8}Q_{xx}l_{12}^2+\tfrac{1}{4}Q_{xt}l_{12}\triangle t+\tfrac{1}{8}Q_{tt}\triangle t^2-\ldots\ldots\ldots$$
$$Q_{22}=Q_M-\tfrac{1}{2}Q_x l_{12}+\tfrac{1}{2}Q_t\triangle t+\tfrac{1}{8}Q_{xx}l_{12}^2-\tfrac{1}{4}Q_{xt}l_{12}\triangle t+\tfrac{1}{8}Q_{tt}\triangle t^2-\ldots\ldots\ldots,$$

where the index x denotes a differential quotient with respect to x and the index t one with respect to t. We deduce approximately

$$(401)\qquad Q_x=\frac{Q_{11}+Q_{21}-Q_{12}-Q_{22}}{2l_{12}}\,.$$

Here the first neglected terms are of the second order in l_{12} and $\triangle t$ as compared to $Q_x l_{12}$. In a similar way we have

$$(402)\qquad h_t=\frac{h_{21}+h_{22}-h_{11}-h_{12}}{2\triangle t}\,.$$

Moreover

$$(403)\qquad \overline{h}=\frac{h_{11}+h_{12}+h_{22}+h_{21}}{4}$$

and $b\approx b(\overline{h})$ may be considered as accurate in the same degree as (401) and (402). Substitution in (201) yields

$$(404)\qquad \tfrac{1}{2}(Q_{11}+Q_{21}-Q_{12}-Q_{22})+B_{12}(\overline{h})\,\frac{1}{2\triangle t}(h_{21}+h_{22}-h_{11}-h_{12})=0,$$

where $B=bl$ denotes the storing surface of the section considered.

In a similar manner we reduce (203) to

$$(405)\qquad \tfrac{1}{2}(H_{11}+H_{21}-H_{12}-H_{22})+\frac{l_{12}}{2g\triangle t}(v_{21}+v_{22}-v_{11}-v_{12})+gH_r=0,$$

where we put

$$(406)\qquad H_r=W_{12}(\overline{h})\,\tfrac{1}{2}(\,|\,Q_{11}\,|\,Q_{22}+\,|\,Q_{12}\,|\,Q_{21}),$$

by W_{12} denoting the resistance of section AB. In (405) the third term of (203) has been neglected.

Similar equations are deduced for the quads 12-22-23-13, 13-23-24-14 etc. The whole set of equations is soluble provided a sufficient number of boundary and initial values is supplied.

The method presented is substantially that of de Vries Broekman. The method of Reineke differs from it in that the derivatives with respect to t are not reduced to differences: they may be determined graphically for instance.

The method of quad-differences has not been applied very often in practice. This is due to the fact that other methods offered greater possibilities.

Firstly the power series or iterative method (4,2) had advantages with regard to the accuracy. An improvement of the accuracy can be attained either by considering shorter sections, or by introducing higher order corrections. The former entails revising the schematization which is very laborius, and for the latter the power series and iterative method are better suited.

Secondly the method of quad-differences is not well suited to deal with single boundary conditions at both ends of a channel which forms a problem of great practical importance. Such conditions are usually introduced in a method of quad-differences by some process of trial and error. On this account a method of cross-differences or a characteristic method is often preferable.

Cross-differences. Often we encounter a problem in which either H or Q is a given constant or a given function of t at each end of a channel. In those cases a method of cross-differences may be appropriate.

We neglect once more the velocity head $v^2/2g$ and treat A as a constant and B and W as constants or given functions of t. We expand H in the environment of point 22. Then:

$$H_{21} = H_{22} + H_x l_{12} + \tfrac{1}{2} H_{xx} l_{12}^2 + \dots\dots\dots$$
$$H_{23} = H_{22} - H_x l_{23} + \tfrac{1}{2} H_{xx} l_{23}^2 - \dots\dots\dots \quad .$$

Suppose that $l_{12} = l_{23}$ or that $l_{23} - l_{12}$ is small compared to l_{12} or l_{23} of the same order as $H_{xx}\, l_{12}$ is small compared to H_x. Then

(407) $$H_x = \frac{H_{21} - H_{23}}{l_{12} + l_{23}}$$

is correct, but for terms of second and higher order. In the same way we find

(408) $$Q_t = \frac{Q_{32} - Q_{12}}{2\triangle t} \; .$$

By introducing the approximation $|Q_{12}|\, Q_{32}$ for $|Q_{22}|\, Q_{22}$ the dynamical equation yields

(409) $$H_{21} - H_{23} + \frac{M_2}{2\triangle t}(Q_{32} - Q_{12}) + W_2\, |Q_{12}|\, Q_{32} = 0.$$

where $M_2 = M_{12} + M_{23}$ and $W_2 = W_{12} + W_{23}$.

In a similar way we apply the continuity equation to point 33:

(410) $$Q_{32} - Q_{34} + \frac{B_3}{2\triangle t}(H_{43} - H_{23}) = 0.$$

Here $B_3 = B_{23} + B_{34}$.

Now suppose $Q(t)$ is given in A and $H(t)$ in F (fig. 3). We suppose moreover that Q_{12}, Q_{14}, H_{23} and H_{25} are given as initial conditions or may be assumed as such. Then we know H_{21}, H_{23} and Q_{12} and compute Q_{32} by (409). In the same way we deduce Q_{34} from H_{23}, H_{25} and Q_{14}. Next we know Q_{32}, Q_{34} and H_{23} and we may compute H_{43} by (410) and furthermore H_{45}.

Then we start anew from the values H_{41}, H_{43} and Q_{32} in order to compute Q_{52} by applying the dynamical equation to the lozenge 32-41-52-43. Proceeding in this way we compute alternately sets of Q values by (409) and sets of H values by (410). The computation gives the vertical tide in B and D and the horizontal tide in C and E. When we want to know the other tides, supplementary computations are necessary for instance with the aid of quad-differences.

Once the boundary and initial conditions given or assumed, the computation proceeds without any trial and error. The influence of a boundary condition is then introduced step by step into the solution along the sides of the lozenges ("lines of influence"). This is decidedly an advantage over a method of quad-differences, in which a boundary value at some instant t_r must be introduced simultaneously in all points of the grid with the abscissa t_r.

The above method, in which the continuity and the dynamical equation are applied alternately, can only be used if each boundary condition is either one in H or one in Q. This restriction can be removed by applying both the continuity and the dynamical equation in every lozenge. It is moreover necessary then to apply these equations in the bordertriangles 16-25-36, 21-32-41 etc. Owing to the lack of symmetry in these triangles, the second order terms cannot be made to cancel, so that second order differences must be introduced in order to maintain the standard of accuracy.

When $v^2/2g$ and the variations of A are no longer negligible, their computation likewise requires the simultaneous application of both equations in every lozenge.

The method of cross-differences has a certain appearance of affinity to the characteristic method (cf Ch. 5), in particular when the section lengths and time intervals are chosen such that the "lines of influence" coincide with the subcharacteristics. This is not essential at all however and, except when B as well as M is constant, it is not even practicable. Let 23-S and 21-S be two subcharacteristics. Then the values of H and Q along the segment 21-23 define the solution in the triangle 23-S-21. Hence, if point 32 lies within this triangle, we may compute the solution in 32 from the data in 21 an 23. When 32 lies outside the triangle, the computation outreaches the propagation of the tidal motion [1]). In that case phantom waves appear in the solution, mainly with the period $4\triangle t$, which tend to grow, in particular near slack water, because then there is but little friction to damp them. The phantom waves remain sufficiently small when the lozenges keep within the subcharacteristic triangles (cf Holsters [33]).

4, 2. Power series and iterative methods.

In order to compute storm tides in the Zuiderzee, the Lorentz Committee [12] developed an integration method by power series. This method was extended to maritime rivers by Dronkers [14] who later converted it into an iterative process [21, 24].

We assume once more a division of intervals of t and sections of x (cf. fig. 3).

1) Physically the influence of a boundary condition proceeds with the characteristic velocity of propagation. For that reason we prefer to retain the name "lines of influence" for the subcharacteristics, and accordingly we prefer to call Holsters' method, a method of cross-differences.

Now consider the motion in the section AB at the instant t_1. The section is supposed to be so short that it is admissible to treat b, m, w and U in (211) and (212) as constants, introducing for them the mean values in the section AB at the instant t_1.

Power series method. Let the origin for x be chosen at the place A and let $H_0(t)$ and $Q_0(t)$ be the functions H and Q in A. Then we can expand H and Q in the environment of the segment 15-16 (fig. 3) by the MacLaurin series:

(411) $$H(x,t) = H_0 + x\left(\frac{\delta H}{\delta x}\right)_{x=0} + \tfrac{1}{2}x^2\left(\frac{\delta^2 H}{\delta x^2}\right)_{x=0} + \ldots\ldots.$$

(412) $$Q(x,t) = Q_0 + x\left(\frac{\delta Q}{\delta x}\right)_{x=0} + \tfrac{1}{2}x^2\left(\frac{\delta^2 Q}{\delta x^2}\right)_{x=0} + \ldots\ldots.$$

When H_0 and Q_0 are known, all the other coefficients can be deduced by means of the differential equations. Then $H(l, t_1)$, and $Q(l, t_1)$ i.e. H and Q in B at the instant t_1, are defined provided the series converge.

The convergence of the expansions in the form (411) and (412) is very difficult to be ascertained, both theoretically and practically. This objection has been overcome by approaching the problem somewhat differently by an iterative process.

Iterative method. We write (211) and (212) in the form

(413) $$\frac{\delta Q}{\delta x} = -b\frac{\delta H}{\delta t} + 2bUQ\frac{\delta Q}{\delta t}$$

(414) $$\frac{\delta H}{\delta x} = -m\frac{\delta Q}{\delta t} \mp wQ^2 + 2bUQ\frac{\delta H}{\delta t},$$

then substitute the approximations $Q = Q_0(t)$ and $H = H_0(t)$ in the right hand members, and integrate from 0 to x. This yields the first subsequent approximations:

(415) $$Q_I = Q_0 - b\dot{H}_0 x + 2bUQ_0\dot{Q}_0 x$$

(416) $$H_I = H_0 - m\dot{Q}_0 x \mp wQ_0^2 x + 2bUQ_0\dot{H}_0 x.$$

Here the points denote derivatives with respect to t.

The second subsequent approximations are found by substituting (415) and (416) in the right hand members of (413) and (414) and integrating once more

(417) $$Q_{II} = Q_I + \tfrac{1}{2}bm\ddot{Q}_0 x^2 \pm bwQ_0\dot{Q}_0 x^2 + Q_s(t,x)$$

(418) $$H_{II} = H_I + \tfrac{1}{2}bm\ddot{H}_0 x^2 \pm bwQ_0\dot{H}_0 x^2 \mp \tfrac{1}{3}b^2 w\dot{H}_0^2 x^3 + H_s(t,x).$$

Here Q_s and H_s denote sets of terms of lower order of magnitude.

Continuation of the process yields Q_{III} and H_{III} etc. Evidently the formulae become more and more involved. As a rule the approximations Q_{II} and H_{II} are sufficient for practical computing. When it is necessary to compute Q_{III} and H_{III}, we must generally as well consider the variations of the coefficients b, m etc.

Generally we have

$$(419) \qquad Q_n = Q_o - b\int_o^x \dot{H}_{n-1}\,dx + 2b\,U\int_o^x Q_{n-1}\dot{Q}_{n-1}\,dx$$

$$(420) \qquad H_n = H_o - m\int_o^x \dot{Q}_{n-1}\,dx + 2bU\int_o^x Q_{n-1}\dot{H}_{n-1}\,dx \mp w\int_o^x Q^2_{n-1}\,dx.$$

These functions are polynomials in x.

It remains to be investigated whether the iterative process is convergent or not for $x \leqslant l$, i.e. whether the series

$$Q = Q_o + \sum_{k=1}^{\infty}(Q_k - Q_{k-1}) \ , \ H = H_o + \sum_{k=1}^{\infty}(H_k - H_{k-1})$$

converge ($Q = \lim\limits_{n \to \infty} Q_n$; $H = \lim\limits_{n \to \infty} H_n$).

Since the terms with U in (413) and (414) are small with respect to the other terms as a rule, we shall leave them out of consideration for the moment. Then it can be proved by induction:

$$(421) \qquad H_{2k} - H_{2k-1} = \sum_{l=2k}^{3(2^k-1)} a_{2k,l}(t)\,x^l \ ; \ Q_{2k} - Q_{2k-1} = \sum_{l=2k}^{2^{k+1}-2} b_{2k,l}(t)\,x^l.$$

$$H_{2k-1} - H_{2k-2} = \sum_{l=2k-1}^{2^{k+1}-3} a_{2k-1,l}(t)\,x^l ; Q_{2k-1} - Q_{2k-2} = \sum_{l=2k-1}^{3.2^{k-1}-2} b_{2k-1,l}(t)\,x^l.$$

The proof of the convergence follows by a more detailed research of the polynomials of (421).

The process is found to be convergent for a group of functions Q_o and H_o. Owing to the quadratic terms in the differential equations, this group is more restricted than if the equations were linear.

In many tidal problems we may assume that all the derivatives of $Q_o(t)$ and $H_o(t)$ are finite with respect to t. In this case there is convergence for limited values of x, depending on the coefficients of the differential equations. We renounce the detailed treatment of this question (cf [24]).

The terms derived by iteration are the same as those appearing in the power series. By their different grouping however, the convergence can more easily be proved. The terms are moreover more easily interpreted physically as contributions to the balance of quantities of water or to the balance of forces and momentum. This facilitates the quantitative appreciation of the terms.

When Q_a and H_a at the upper end of a section are known, we can compute Q_b and H_b at the lower end, or inversily we compute Q_a and H_a from Q_b and H_b.

Hence if we know Q and H at the inlet of an estuary or at the mouth of a river, we can compute these quantities up the estuary or river from section to section as far as the closed end or until the range is so small that further calculation serves no purpose.

However in various practical applications we do not know $Q\ (t)$ at the inlet. We can then arrive at this Q by trial and error, checking by means of a boundary condition at the landward extremity which is usually known.

Other applications will be discussed in the subsequent sections.

4, 3. Computation of currents from vertical tides; single sections.

In general it is simpler to record water levels than currents. So the problem arises to calculate the discharges and velocities in a section as a function of time when the water levels at both ends are known. This can be performed by means of the formulae of the preceding section.

We assume the Chézy coefficient C to be known. Let the section be so short (in the Dutch practice usually sections of 10 km or less are considered) that the second iteration (418) is sufficient. Putting for x the length l of the section, H_{II} becomes the known vertical tide at one end whereas H_o is the known tide at the other end of the section. Thus (418) becomes a nonlinear differential equation in Q_o of the first order.

Notwithstanding its nonlinear character it is very simple to determine Q_o as a function of time numerically, when we know or may assume an initial condition for Q_o (cf. 2,3). We may do so by converting the derivatives with respect to t into quotients of finite differences. Then we compute Q_o from interval to interval starting from $Q_o\ (t_1)$ at the instant t_1. In practice, intervals of about a quarter of an hour usually will do.

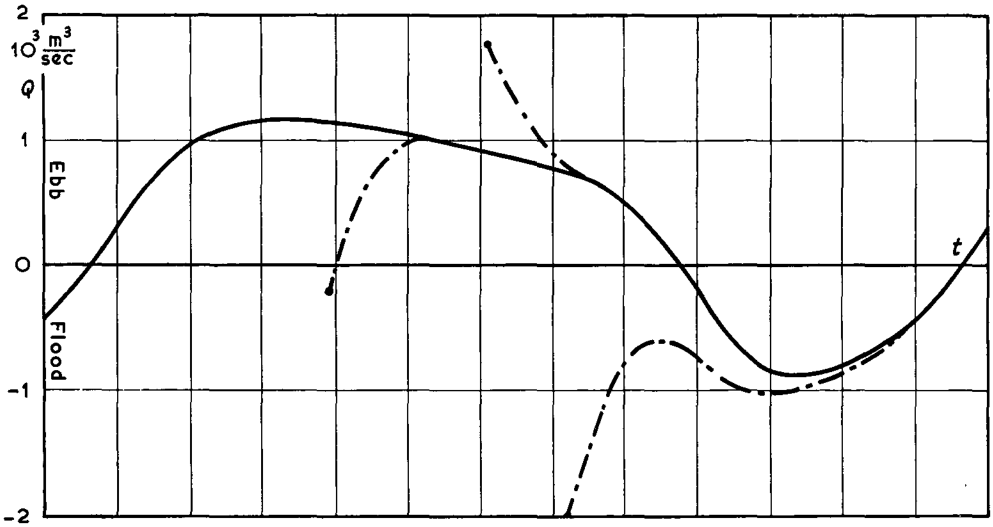

Fig. 4. Computation of the discharge curve starting from different initial conditions.

When the vertical tides H_0 an H_1 are periodic it can be shown that the differential equation in Q_0 has one unique periodic solution. This solution is stable and hence, if we start from an arbitrary value $Q_0(t_1)$, the integral curve will approach the periodic solution asymptotically with increasing time (cf fig. 4).

If we are so fortunate as to know the local vertical tides along the estuary at distances of about 10 km or less, we are able to check the schematization used. Then we calculate the discharges at the beginning of each section by the above method. These discharges however must also satisfy the equation of continuity (417). This provides a check of the schematization of the estuary and of the Chézy coefficient.

4, 4. Computation of currents from vertical tides; combinations of sections.

The above method is not well applicable when we only know the vertical tides at the ends of a channel with a length largely exceeding 10 km. Then we may consider the following method:

Suppose the channel is so long that it should be divided into two sections. In each of these sections we adopt mean values for m, b and w as outlined in 4,2. We denote by Q_1, H_1 the discharge and head at the beginning of the first section, by Q_{12}, H_{12} halfway the section, and so on as fig. 5 shows.

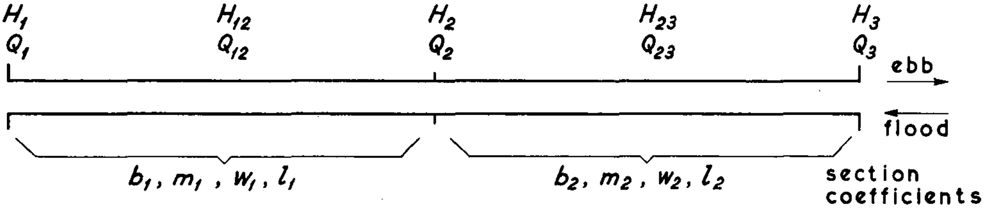

Fig. 5. Combination of two sections.

Now we apply (416) to the centre of a section omitting the secondary terms involving U, substitute $x = -\frac{1}{2}l$ and $x = \frac{1}{2}l$ and then deduce

(422) a) $H_2 = H_1 \mp w_1 Q_{12}^2 l_1 - m_1 \dot{Q}_{12} l_1$ b) $H_3 = H_2 \mp w_2 Q_{23}^2 l_2 - m_2 \dot{Q}_{23} l_2$.

Moreover we may put

(423) a) $Q_{12} = Q_2 + \frac{1}{2} b_1 \dot{H}_{12} l_1$ b) $Q_{23} = Q_2 - \frac{1}{2} b_2 \dot{H}_{23} l_2$.

Now we put approximately

$$\dot{H}_{12} = \tfrac{1}{2}(\dot{H}_1 + \dot{H}_2) = \tfrac{3}{4}\dot{H}_1 + \tfrac{1}{4}\dot{H}_3 \; ; \qquad \dot{H}_{23} = \tfrac{1}{4}\dot{H}_1 + \tfrac{3}{4}\dot{H}_3 ,$$

so that

$$Q_{12} = Q_2 + \tfrac{1}{8} b_1 (3\dot{H}_1 + \dot{H}_3) l_1 \; ; \qquad Q_{23} = Q_2 - \tfrac{1}{8} b_2 (\dot{H}_1 + 3\dot{H}_3) l_2 .$$

Substituting in (422) we find after some calculation:

$$H_3 - H_1 = \mp (w_1 l_1 + w_2 l_2)\, Q_2^2 - (m_1 l_1 + m_2 l_2)\dot{Q}_2 + \tag{424}$$
$$\mp \tfrac{1}{4} \{ w_1 l_1^2 b_1 (3\dot{H}_1 + \dot{H}_3) - w_2 l_2^2 b_2 (\dot{H}_1 + 3\dot{H}_3) \}\, Q_2 +$$
$$\mp \tfrac{1}{64} \{ w_1 l_1^3 b_1^2 (3\dot{H}_1 + \dot{H}_3)^2 + w_2 l_2^3 b_2^2 (\dot{H}_1 + 3\dot{H}_3)^2 \} +$$
$$- \tfrac{1}{8} \{ m_1 l_1^2 b_1 (3\ddot{H}_1 + \ddot{H}_3) - m_2 l_2^2 b_2 (\ddot{H}_1 + 3\ddot{H}_3) \}.$$

Thus we have derived a differential equation for Q_2, which we may solve as treated in 4,3. After calculating Q_1 we determine

$$Q_1 = Q_2 + b_1 \dot{H}_{12}\, l_1 \tag{425}$$

as a first approximation of Q_1.

Starting from Q_1 and H_1 we compute Q_2 and H_2, and then Q_3 and H_3 with the aid of (417) and (418). Now H_3 has to be identical with the known function H_3, but there will generally be deviations. Then we can determine a closer approximation for Q_1 by putting for $\dot{H}_{12}$ and $\dot{H}_{23}$ the functions computed from Q_1 and H_1. This yields a formula analogous to (424) from which we determine Q_1 again.

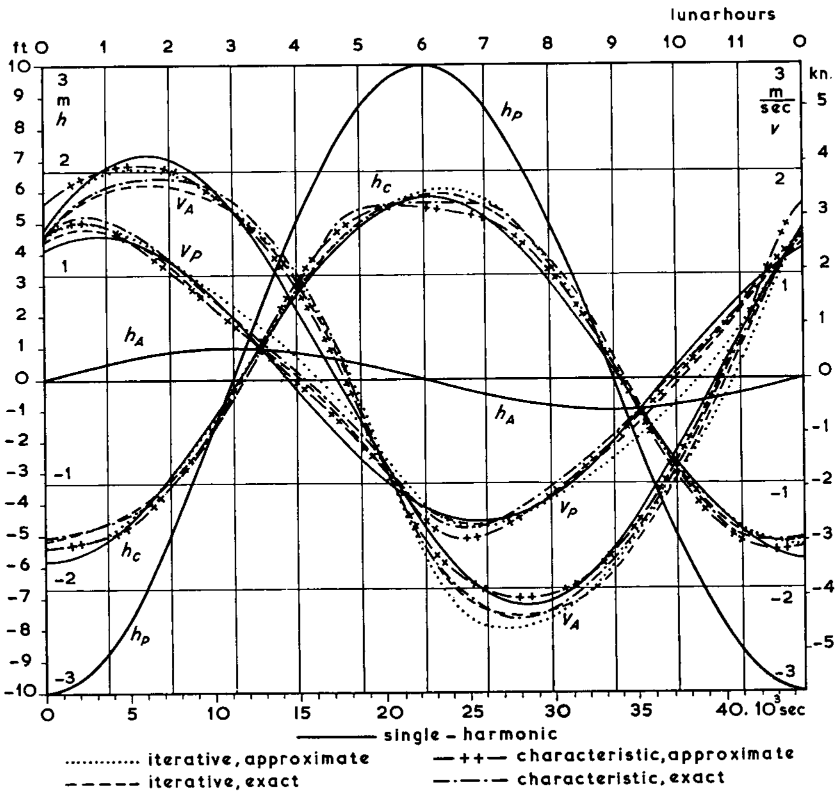

Fig. 6. Panama sea level canal. Computations by various methods.

We can apply this method for relatively large channels. In the Dutch estuaries the total length may be up to 30 km.

In order to show an example of the preceding method, we have calculated the tidal movement in an open unregulated Panama sea level canal using the schematization of Lamoen [25] (cf 4,1). We have divided the canal in 4 sections and computed a first approximation by the above method. Then refinements have been computed by (417) and (418). Both solutions are represented in fig. 6.

4, 5. Application to the planning of the enclosure of a tidal river.

As stated in 1,1, tidal computations form a valuable support in the planning of the enclosure of a tidal river. In order to illustrate this, we shall briefly discuss the computations with regard to the enclosure of the Brielse Maas, which is represented schematically in fig. 7.

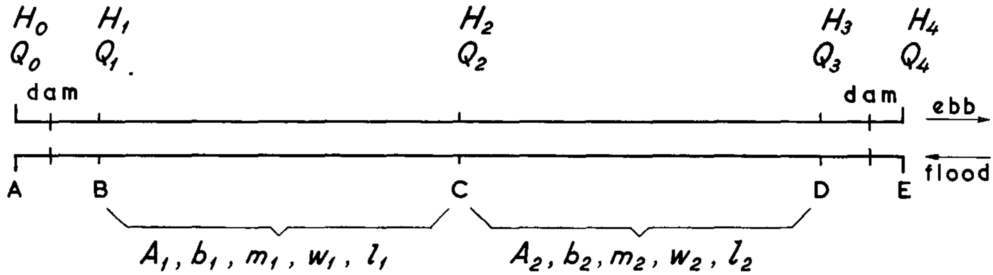

Fig. 7. Scheme of river Brielsemaas.

Between A and B the river had to be closed by a dam, and likewise between D and E. C is almost halfway B and D. The sections BC and CD are each about 11 km long. The vertical tides at A and E, H_0 and H_4 are known. The distances A-B and D-E are so small that we may put

$$Q_0 = Q_1 \text{ and } Q_3 = Q_4.$$

As the construction of the dam between A and B progresses, the flow passes through a narrowing gap. Between C and D the situation is simular. In the gap A-B there is a loss of head

(426) $$H_0 - H_1 = \pm W_1 Q_1^2, \qquad \begin{array}{l}(+\text{ for ebb})\\(-\text{ for flood})\end{array}$$

where

$$W_1 = \frac{\eta_1}{2g}\left(\frac{1}{A_g} - \frac{1}{A_1}\right)^2 \quad \text{or} \quad W_1 = \frac{\eta_2}{2g}\left(\frac{1}{A_g} - \frac{1}{A_0}\right)^2$$

in case of ebb or flood respectively. Here A_g is the cross-sectional area of the gap, A_0 that area above, and A_1 that below the gap, whereas η_1 and η_2 are coefficients of the gap.

Likewise

(427) $$H_3 - H_4 = \pm W_3 Q_3^2$$

holds good for the gap between C and D.

The basin of the Brielse Maas, represented by the sections BC and CD, is treated in the way of 4,4. In (426) and (427) we substitute (425) and an analogous expression for Q_3. Hence we obtain three equations, (424) (426) and (427), by which we may determine Q_2, H_1 and H_3.

After elimination of H_1 and H_3 with the aid of (426) and (427) we get a nonlinear differential equation for Q_2 which may be solved in a similar manner as treated in 4,3 and 4,4.

The following questions concerning the closing of the two gaps were put:

1. Which gap has to be closed first?
2. Is it possible to narrow the gaps in coordination with each other in such a way that the closing of one of the gaps would become easier?
3. Which are the values of the velocities in the gaps during the process of narrowing? Both the maximum velocities during a tide, and the slack water conditions are important from the engineering point of view.

The results of the tidal calculations showed that it was preferable to narrow first the gap at the seaside. Then the velocities in the other gap (riverside) would decrease because the penetration of the tide into the channel is obstructed. In fact the gap at the riverside could be closed without difficulty. After that the gap at the seaside had to be closed, which was effected by sinking a large pontoon at slack tide.

5. INTEGRATION ALONG CHARACTERISTICS

The principle of the characteristics and their bearing on the phenomenon of propagation can be most clearly discussed in connection with linear equations without resistance (5, 1). Next the amendments to be made in order to deal with the resistance are expounded (5,2). Furthermore the variability of the velocities of propagation is dealt with (5,3). Finally shock wave conditions are considered (5,4).

5, 1. Elementary theory of propagation.

We start from (209) and (210) where we consider b and m as constants. In this section we make moreover abstraction from the resistance:

$$\frac{\delta Q}{\delta x} + b\frac{\delta H}{\delta t} = 0 \tag{501}$$

$$\frac{\delta H}{\delta x} + m\frac{\delta Q}{\delta t} = 0. \tag{502}$$

It be observed that, according to the disregard of $v^2/2g$ presumed in deriving (209) and (210), H may be interpreted arbitrarily as the water level or as the total head.

The tidal motion is mathematically described by H and Q as functions of t and x. This can be represented graphically by using a HQ-diagram in connection with a tx-diagram (fig. 8).

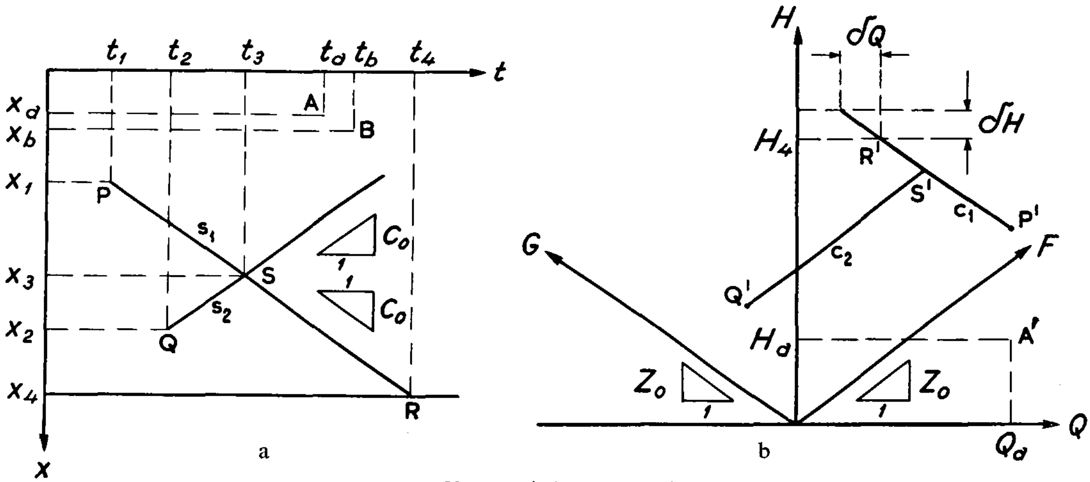

Fig. 8. Characteristic construction diagrams.
a. Diagram of itineraries, b. Diagram of states of motion.

Let us consider the point t_a, x_a in the tx-diagram (A in fig. 8). This means that we fix our attention to the place x_a at the instant t_a. Let H_a be the head and Q_a the discharge in this place at that instant. Then the point (H_a, Q_a) in the HQ-diagram (A′ in fig. 8) representing the *state of motion* in x_a at t_a, is associated with the point (t_a, x_a) in the tx-diagram (A).

Any function F of H and Q may be interpreted as a property of the state of motion. If a definite value F_a of such a function F is observed at the instant t_a in a place x_a, and shortly after at the instant t_b in a slightly further place x_b, we shall say that the property represented by F is *propagated* from x_a to x_b during the interval t_a to t_b.

The further mathematical development of this idea consists in trying to deduce from (501) and (502) one or more equations of the form

$$\frac{\delta F}{\delta t} + c\frac{\delta F}{\delta x} = 0.$$

For the elaboration we may refer to [28] Ch 2. As a result of the analysis there appear to be two such equations which we can find by adding $\frac{1}{2} Z_0 c_0$ times (501) to $\frac{1}{2} c_0$ or to $-\frac{1}{2} c_0$ times (502). The two functions which are propagated in the above sense are

(503) a) $F = \frac{1}{2} H + \frac{1}{2} Z_0 Q$ and b) $G = \frac{1}{2} H - \frac{1}{2} Z_0 Q$,

satisfying the equations

(504) a) $\dfrac{\delta F}{\delta t} + c_0 \dfrac{\delta F}{\delta x} = 0$ and b) $\dfrac{\delta G}{\delta t} - c_0 \dfrac{\delta G}{\delta x} = 0,$

where

(505) a) $c_0 = 1/\sqrt{bm}$ b) $Z_0 = 1/Y_0 = \sqrt{m/b}.$

In a geometric point moving in the positive x-sense with the velocity $dx/dt = c_0$, we have

(506) $$dF = \frac{\delta F}{\delta t} dt + \frac{\delta F}{\delta x} dx = \left(\frac{\delta F}{\delta t} + \frac{\delta F}{\delta x} c_0\right) dt,$$

which is zero by virtue of (504a). Hence F preserves its value in the moving point. This point, which we shall call a *wave point*, can be said to convey that particular value of F. In a similar way a wave point moving in the negative sense with the velocity $dx/dt = -c_0$, conveys a particular value of G.

The functions F and G may be considered as new coordinates in the HQ plane. Hence a state of motion is as well defined by F and G as by H and Q. We shall call F and G the characteristic *wave components* of the tidal motion. The values of these components are conveyed by the wave points. Thus the component F is propagated in the positive sense and the component G in the negative sense. The *velocity of propagation* is c_0 or $-c_0$.

The itineraries of the wave points are represented by straight lines in the tx-diagram, called *subcharacteristics*, with slopes c_0 to 1 or $-c_0$ to 1. The HQ-points associated with the points of a particular subcharacteristic, all correspond to the same value of F or G, and hence are situated on straight lines in the HQ-diagram, called *contra-subcharacteristics*. When the slope of the subcharacteristic is c_0 to 1, then by (503a) the slope of the associated contra-subcharacteristic is $-Z_0$ to 1 (compare s_2 and c_2 in fig. 8). When the slope of the former is $-c_0$ to 1, then by (503b) that of the latter is Z_0 to 1 (s_2 and c_2 in fig. 8). A subcharacteristic and the associated contra-subcharacteristic together will be referred to as a *characteristic*. If the solution

$H(t,x), Q(t,x)$ is represented by an integral surface in a $HQtx$-hyperspace, the characteristics are curves on that surface and the subcharacteristics and contra-subcharacteristics are projections of the characteristics.

Let us consider a travelling wave invading a state of rest (fig. 9). The water surface in rest having been adopted as zero level, we have $H = 0$ and $Q = 0$ and hence $F = 0$ and $G = 0$ in the undisturbed region. Any wave point P moving with the wave and conveying a particular value F_p of F, meets wave points coming out of the region of rest and thence conveying the value $G = 0$. So we have from (503a) and b)

(507) a) $H_p = F_p$ and b) $H_p = Z_0 Q_p$

in the wave point P independent of t. This means firstly that there is a definite relation (507-b) between H and Q and secondly that the whole configuration of elevations and depressions and the associated currents, displaces with the velocity c_0 of the wave points in virtue of (504-a).

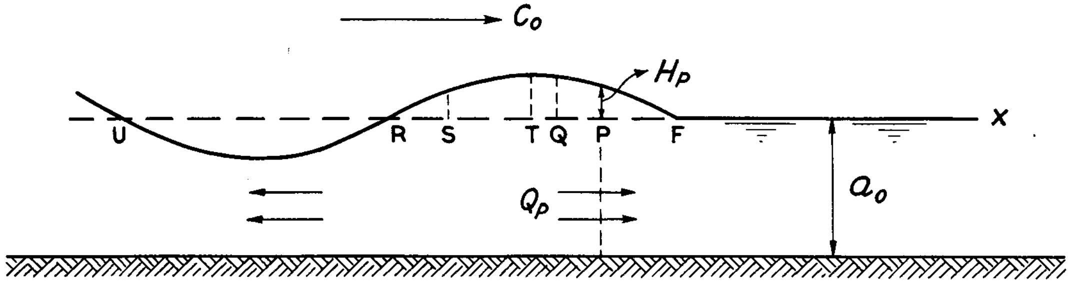

Fig. 9. Travelling (tidal) wave. Vertical dimensions greatly exaggerated compared to horizontal dimensions. Wave height exaggerated compared to depth.

The elevations of the water level are attended by currents in the sense of the propagation and the depressions by currents in the opposed sense. The points R and U where the original level is attained, also mark the places of slack water (cf (507b)).

It be observed that in a canal with a rectangular cross-section of width b and depth a_0, we have $m = 1/ga_0b$ and hence

$$c_0 = \sqrt{ga_0},$$

a well-known formula for the velocity of propagation of a travelling wave.

Apparently the travelling wave character of the water motion is associated with the wave points moving with the velocity c_0, which we shall call the *active* (manifest, cf [28] Ch. 3, sect. 11) wave points. The wave points moving in the other sense and conveying the rest value $G = 0$ are *inactive* (latent).

In the state of rest all wave points are inactive. This does not imply however that those wave points are at rest themselves.

Now we consider a wave motion in which the wave points moving in both senses are active. In order to define the line of thought, suppose we know the state of motion in a place x_1 at an instant t_1 and likewise in x_2 at t_2. So we consider the points P and Q in fig. 8a. Let P′ and Q′ in the *HQ*-diagram represent the associated states of motion. Now we consider a wave point moving with the velocity c_0 in the positive sense, passing by the place x_1 at the instant t_1 on its way to the place x_2. The itinerary of this wave point is the sub-characteristic s_1. In the same way s_2 is the itinerary of an other wave point moving with the velocity $-c_0$ in the negative sense and passing by the place x_2 at the instant t_2.

In the *tx*-diagram we see that the two wave points meet in a place x_3 at the instant t_3. This event is represented by the intersection S of the subcharacteristics s_1 and s_2. The associated state of motion is easily constructed in the *HQ*-diagram. For the *HQ*-point S′ of this state of motion must lie as well on the contra-subcharacteristic c_1 associated with the first wave point, as on the contra-subcharacteristic c_2 associated with the second wave point.

Now suppose the head H in the place x_4 is controlled by a boundary condition. This place for instance is the inlet of the estuary. Then the head H_4 at the instant t_4, at which the wave point first considered above arrives in the place x_4, is given. The *tx*-point R represents the arrival of the wave point. The *HQ*-point R′ representing the associated state of motion, must have the ordinate H_4. The point R′ must moreover lie on the contra-subcharacteristic c_1. These two conditions define R′ entirely.

If the boundary condition controls Q or a definite function of H and Q, the state of motion point R′ is defined in a similar way.

Let now the head in the place x_4 be suddenly varied at the very instant of the arrival of the wave point so that H becomes $H_4 + \delta H_4$. The *HQ*-point representing the state of motion immediately after the variation of head, cannot but lie on c_1 as well as R′. This means that the variation of head δH_4 is attented by a variation $\delta Q_4 = -\delta H_4/Z_0$ (cf (503a)).

Any sudden variation of head imposed on the canal $(x < x_4)$ provokes a proportional reaction in the discharge and inversely. The increase of head per unit decrease of discharge, Z_0, is called the characteristic coefficient of *wave impediment.* The reciprocal Y_0 is called the *wave admission.*

In the same way it is argued that sudden variations imposed in the other sense (say to a canal with $x > x_4$) provoke likewise reactions such that $\delta H = Z_0 \delta Q$.

The above constructions illustrate some of the main procedures applied in the characteristic approach of tidal problems. Other such procedures serve to deal with reflections at widenings or narrowings of a channel, at junctions of channels etc.

Before we set out to describe more systematically how a tidal computation by characteristics is performed, we must first pay attention to the resistance in view of its practical importance.

5, 2. Influence of frictional resistance on propagation.

We add again $\frac{1}{2} Z_0 c_0$ times (209) to $\frac{1}{2}\, c_0$ or $-\frac{1}{2}\, c_0$ times (210) so that instead of (504):

(508) a) $\frac{\delta F}{\delta t} + c_0 \frac{\delta F}{\delta x} + \frac{1}{2} c_0 w \mid Q \mid Q = 0$; b) $\frac{\delta G}{\delta t} - c_0 \frac{\delta G}{\delta x} - \frac{1}{2} c_0 w \mid Q \mid Q = 0$,

where F and G are still defined by (503). The term $w \mid Q \mid Q$ may be considered as a function of F and G. If w is supposed constant, then by (503)

$$w \mid Q \mid Q = w Y_0^2 \mid F - G \mid (F - G)$$

formulates the resistance as function of F and G.

The function F in a wave point moving with the velocity c_0 now no longer preserves its value. Substitution from (508-a) in (506) yields

(509) $$\mathrm{d}F = -\tfrac{1}{2} c_0 w \mid Q \mid Q \, dt.$$

Hence F decreases gradually if Q is positive and increases if Q is negative. Similarly G in a wave point moving with the velocity $-c_0$ increases if Q is positive and decreases if Q is negative.

By the above mathematical arrangement it is possible to explain the behaviour of tidal and similar waves in terms of propagation and attenuation associated therewith. The following may illustrate this:

Consider once more the wave of fig. 9. In the region of rest we still have $F = 0$ and $G = 0$. Now a wave point emerging from that region to meet the wave, conveys the value $G = 0$ until it encouters the foot F of the wave. Then it enters a region where $Q > 0$. Hence the value of G will begin to increase and G becomes positive. So the encountering wave point is activated. Since generally by virute of (503-b)

(510) $$Z_0 Q = H - 2G,$$

the currents will be weaker than if there were no resistance.

By consequence of (510) the place of slack water will be found at a positive elevation $H = 2G \geqslant 0$, say in S. At this place, G in the receding wave point reaches its maximum value and G will begin to decrease behind S.

The strongest currents are by virtue of (510) found where

$$\frac{\delta H}{\delta x} = 2 \frac{\delta G}{\delta x}.$$

Since $\delta G/\delta x$ is negative between F and S, the place searched for must have $\delta H/\delta x$ negative as well. Hence the maximum flow no longer coincides with the top T, but with a more forward point Q.

The assumption of a region of rest ahead of the wave is not essential for the conclusions, although for other assumptions the line of argument is slightly more complicated. Hence it is explained why in a sine wave subject to resistance, the horizontal tide has a phase lead with respect to the vertical tide (cf 3,1).

When the line of thought is followed up further it can be explained that the displacement of the top of a travelling wave lags behind the wave point, so that the phase velocity appears to be less than c_0.

Now consider once more the estuary AF to which fig. 3 refers. The vertical tide H

is supposed to be known as function of t at the inlet F and Q is known at the end A. Moreover we suppose that H and Q in the whole estuary at the instant t_0 are known as initial conditions or may be assumed as such.

We divide the estuary into a number of sections of equal time of propagation:

$$\tau_0 = l/c_0 = \sqrt{BM}.$$

Let ABCDEF in fig. 3 represent such a division. Moreover the interval $\triangle t$ of the time division is put equal to τ_0. Then the sides of the lozenges are subcharacteristics.

The computation by characteristics now proceeds as follows: From the states of motion associated with the tx-points 12 and 14 we deduce the state of motion associated with 23. Likewise we proceed from 14 and 16 to 25. From 25 we proceed to 36 and define H there with the aid of the given value of Q. Then we go from 12 to 21 and use the boundary condition in F. Next we proceed from 21 and 23 to 32 and from 23 and 25 to 34. Then we go to 41 and so on. The order in which the points are treated might be chosen somewhat differently.

If there were no or a negligible resistance the procedures represented in fig. 8 might be used. As the resistance in tidal motions is quite appreciable as a rule, we must modify the constructions as will be discussed below.

Let P′ in fig. 10a represent the state of motion associated with the tx-point 43 in fig. 3 and let likewise Q′ correspond to 41. The state of motion associated with 52 and represented by the HQ-point S′, must be determined by constructing the contra-subcharacteristics associated with 43-52 and 41-52.

Consider the wave point of which 43-52 represents the itinerary. It encounters series of states of motion beginning with that represented by P′ and ending with that represented by S′. The factor $|Q|Q$ on its journey varies from $|Q_P|Q_P$ to $|Q_S|Q_S$. Let $\overline{|Q|Q}$ denote the average of $|Q|Q$ during this journey: Then we deduce from (509)

(511) $$F_S - F_P = -\tfrac{1}{2}c_0 w\,\overline{|Q|Q}(t_1 - t_0) = -\tfrac{1}{2}W_{23}\,\overline{|Q|Q}.$$

Here W_{23} is the resistance of the section DE. For $c_0(t_1 - t_0)$ is equal to the length l_{23} of that section and $W_{23} = wl_{23}$.

Substituting from (503) in (511) yields

(512) a) $$(H_S - H_P + W_{23}\overline{|Q|Q}) + Z_0(Q_S - Q_P) = 0.$$

This means that the point S′ with the coordinates H_S and Q_S must lie on a line k sloping at Z_0 to 1 and through a point D which lies at the distance $W_{23}\overline{|Q|Q}$ below P′. This line can be constructed when we make an estimation of $\overline{|Q|Q}$.

In a similar manner we have

(512) b) $$(H_S - H_Q - W_{12}\overline{|Q|Q}) - Z_0(Q_S - Q_Q) = 0$$

and we can construct a line m in connection with the wave point of which 41-52 is the itinerary. The point S′ is then found at the intersection of k and m.

The contra-subcharacteristics are not the lines k and m, but the lines P′ S′ and Q′ S′.

In fig. 10a we have also represented the construction for the state of motion point R′ associated with 61 (fig. 3). Here the datum H_R is introduced.

The estimation of $|Q|Q$ can be checked after the construction of S′, and if necessary the construction is repeated with a new estimate. It appears, however, that these estimations often require a relatively great amount of trial and error labour. In view of its practical importance we go somewhat deeper into the question:

We approximate the variation of Q along the subcharacteristic 43-52 by

$$Q = Q_P + (Q_S - Q_P)\, x / l_{23},$$

for a moment adopting the place D as origin for x. Then, if Q_p and Q_s have equal signs,

$$(513)\quad \overline{S} = \overline{|Q|Q} = \frac{1}{l_{23}} \int_0^{l_{23}} |Q|Q = \pm(\tfrac{1}{3} Q_P^2 + \tfrac{1}{3} Q_P Q_S + \tfrac{1}{3} Q_S^2).$$

This is a function of Q_P which is known and Q_S which is unknown and ought to be estimated. Unfortunately $\overline{S}$ is very sensitive to errors in the estimation of Q_S and this is the root of the difficulties expounded above.

In order to substantially eliminate this drawback, we rewrite (513) as follows:

$$\overline{S} = |Q_P|\, Q_P + Q_m(Q_S - Q_P) \quad \text{where} \quad Q_m = |\tfrac{2}{3} Q_P + \tfrac{1}{3} Q_S|\,.$$

The estimation of Q_S is now only used to determine the factor Q_m which is relatively little sensitive. Then (512) may be reduced to

$$(514)\qquad (H_S - H_P + H_{rP}) + Z(Q_S - Q_P) = 0,$$

where $\quad H_{rP} = W_{23}\, |Q_P|\, Q_P \quad$ and $\quad Z = Z_0 + W_{23}\, Q_m\,.$

Since Q_m is determined by estimation, (514) constitutes a linear equation in Q_S. It is constructively interpreted in the way described already; the distance P′D becomes H_{rP} and the slope of k becomes Z to 1.

When Q_P and Q_S have different signs, we may introduce the approximating formulae

$$Q_m = \tfrac{2}{3}\, |Q_S| \quad \text{and} \quad H_{rP} = \tfrac{1}{3}\, W_{23}\, |Q_P - Q_S|\, (Q_P - Q_S),$$

where we make use of an estimation for Q_S, and further proceed as described above. The justification of this procedure would demand a disproportionate space and hence is omitted here.

A slightly different method with fixed subcharacteristics was applied by Lamoen to the Panama canal [25]. The canal was divided into six sections. The results of Lamoen's computation are represented in fig. 6.

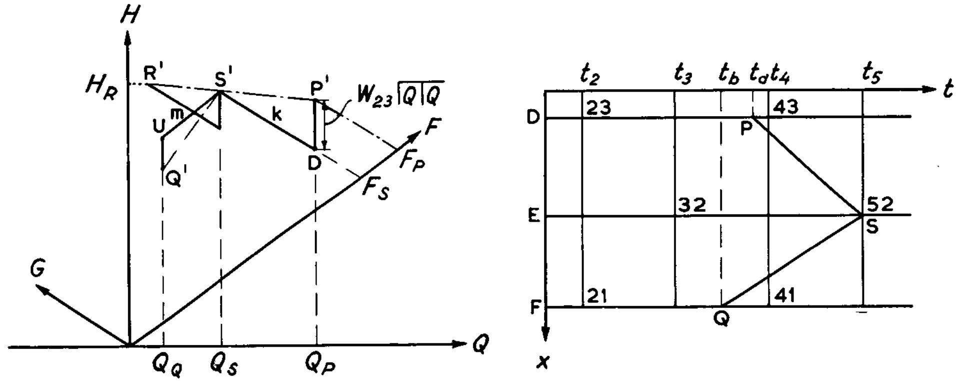

Fig. 10. a. Construction to account for resistance. b. Construction to account for variations of velocities of propagation.

5, 3. Exact computation by variable velocities of propagation.

We apply the characteristic transformation (cf [28] Ch 2) to the exact equations (207) and (208). This means that we should add $Z^{\mp}$ times (207) to (208) where $Z^{\mp}$ is a factor to be chosen in such a way that we arrive at equations of the form

$$\frac{\delta H}{\delta x} + \frac{1}{c^{\pm}}\frac{\delta H}{\delta t} + w \mid Q \mid Q = Z^{\mp}\left[\frac{\delta Q}{\delta x} + \frac{1}{c^{\pm}}\frac{\delta Q}{\delta t}\right]. \tag{515}$$

Elaboration yields

$$\text{(516)}\quad \text{a)}\quad c^{+} = v + c_o \sqrt{1 + \frac{b - b_s}{b_s}\cdot\frac{v^2}{v_c^2}} \quad\text{and b)}\quad c^{-} = v - c_o \sqrt{1 + \frac{b - b_s}{b}\cdot\frac{v^2}{v_c^2}}$$

$$\text{(517)}\quad \text{a)}\quad Z^{+} = Z_o \sqrt{1 + \frac{b - b_s}{b_s}\cdot\frac{v^2}{v_c^2}} \quad\text{and}\quad \text{b)}\quad Z^{-} = -Z_o \sqrt{1 + \frac{b - b_s}{b}\cdot\frac{v^2}{v_c^2}}.$$

Here c_o stands for $1/\sqrt{bm}$ and Z_o for $\sqrt{m/b}$ as in (505); however, now m and b and hence c_o and Z_o are not constant but depending on the state of motion, i.e. on H and Q. Furthermore v denotes the velocity of flow $v = Q/A$ and v_c is the *critical velocity of flow* defined by

$$v_c = \sqrt{gA/b_s} = c_o\sqrt{b/b_s}, \tag{518}$$

to which we return further below.

It follows from (515) that in a wave point moving with the velocity $dx / dt = c^{+}$, the states of motion satisfy the differential equation

$$dH + w \mid Q \mid Q\, dx = Z^{-} dQ \qquad [\text{if } dx = c^{+} dt]. \tag{519—a}$$

Likewise

$$dH + w \mid Q \mid Q\, dx = Z^{+} dQ \qquad [\text{if } dx = c^{-} dt] \tag{519—b}$$

holds good in a wave point moving with the velocity c^{-}. We might bring (519—a) or b) in a form like (509) by introducing the variables F and G, defined as functions of H and Q by

$$\frac{\delta F}{\delta Q} + Z^- \frac{\delta F}{\delta H} = 0 \qquad \text{or} \qquad \frac{\delta G}{\delta Q} + Z^+ \frac{\delta G}{\delta H} = 0.$$

However, the solution of these equations, in which Z^+ and Z^- are functions of H and Q and moreover of x, cannot be generally formulated. It moreover serves no practical purpose since we can as well operate directly with (519 a) and b).

The velocity of propagation c^+ or c^- can be either positive or negative depending on the velocity of flow. The critical value of the velocity of flow is v_c defined by (518). When $-v_c < v < v_c$, the flow is *subcritical* (flowing water) and $c^+ > 0$ and $c^- < 0$. When $v > v_c$ or $v < -v_c$, the flow is *supercritical* (running or shooting water) and $c^+ > c^- > 0$ or $c^- < c^+ < 0$. The distinction is of great practical consequence for the influence of boundary conditions on the flow. We shall not go further into this question and confine ourselves to subcritical flow (cf [28] Ch 3, sect 212).

Tidal motions usually are largely subcritical, i.e. v is small compared to v_c. We may then be justified in approximating by

(520) a) $c^{\pm} = v \pm c_o$ and b) $Z^{\pm} = \pm Z_o$

and by determining c_o, Z_0 and W as functions of H by neglecting $v^2/2g$.

We now consider again the estuary AF mentioned before. A division into sections of approximately equal time of propagation is established. Furthermore suppose for the moment that the tidal curves are smooth continuous functions of time. We can then perform the computation according to a grid as shown by fig. 3, in which the lozenges keep within the subcharacteristic triangles. We shall not describe this systematically but only give an illustrative example:

Suppose the computation has been completed as far as the instant t_4, so that we know the states of motion associated with the tx-points 21, 23, 32, 41, 43 etc. Now we proceed to 52 by considering the two wave points meeting each other in the place E at the instant t_5 (tx-point S in fig. 10b). Let t_a be the instant at which the descending wave point passed by D (tx-point P in fig. 10b) and t_b the instant at which the ascending wave point left F (tx-point Q in fig. 10b).

The subcharacteristic representing the itinerary of the descending wave point goes through P and S. The straight line PS which is a chord of this subcharacteristic, is constructed as follows:

Let (H_P, Q_P) and (H_S, Q_S) be the states of motion associated with P and S. Then we put

$$Q_g = \tfrac{1}{2} Q_P + \tfrac{1}{2} Q_S \qquad \text{and} \qquad H_g = \tfrac{1}{2} H_P + \tfrac{1}{2} H_S$$

and derive $c_g^+ = Q_g / A + c_o(H_g)$ from them as an approximation for the average velocity of the descending wave point. This will be sufficiently accurate provided the sections of the estuary are not too great. Now we estimate H_P, Q_P, H_S and Q_S, compute c_g^+, and construct P by drawing PS through S by the slope c_g^+ to 1. Then the estimations of H_P and Q_P can be checked by interpolating between the states of motion in D at the instants t_2 and t_4 which were supposed to be known. The check of H_S and Q_S follows later.

In a similar way we construct SQ and interpolate for H_Q and Q_Q. Then the HQ-points P′ and Q′ associated with P and Q are known and S′ is constructed as indicated in fig. 10a. This provides the check for the estimation of H_S and Q_S. If necessary the whole set of constructions is repeated.

The interpolations mentioned may be performed graphically by two auxiliary diagrams, an Ht- and a Qt-diagram.

We conclude this section by a few remarks:

The approximations and estimations introduced above do not form an essential feature of the method. They usually meet the requirements in the Dutch practice, where we have channels of some 5 or 10 m depth and where we can operate with sections of 5 to 10 km with times of propagation of some 10 or 15 min. In different situations modifications in the performance should be considered. The method could be applied practically unmodified to the Panama sea level canal which was divided into 4 sections each with a time of propagation of about 20 min. The results are shown in fig. 6.

A fixed grid like that of fig. 3 can be used profitably when the tidal motion shows a gradual trend. When there are acute bends in the tidal curves, e.g. in case of manipulations with locks etc., the fixed grid must be abandoned and the characteristics marking the propagation of the sharp details should be followed uninterruptedly.

Instead of graphical constructions, a numerical procedure may be used. This may be preferable when computing machines are used. From the engineer's point of view the graphical constructions have the advantage of helping to visualize the procedure.

5, 4. Jumps and bores.

The bore is a tidal phenomenon observed in rather shallow estuaries and rivers with a great tidal range, which occurs in particular when the estuary or river is funnel-shaped. The bore comes into being when higher parts of the rising tide front of the penetrating tidal wave tend to overtake lower parts. This is put into evidence by the tx -diagram where the concurrent subcharacteristics of the wave points in the rising tide front are intersecting.

The bore is a hydraulic jump which is not fixed in place but travelling up the river. Similar mobile jumps may arise from too abrupt operations with locks etc.

Although in these mobile jumps the vertical accelerations are of essential influence, it is still possible to compute a tide with a bore or other jump as a long wave, provided we account for the jump as a discontinuity in the long wave localised in a definite moving point comparable to a wave point. Doing so is justified since the length of the bore is as a rule very small as compared to the length of the tidal wave.

A rigorous treatment of the mobile jump by rather simple formulae can be given if the cross-section of the channel is rectangular (cf [28] Ch 12 sect 31). The formulae for an arbitrary cross-section with a storing width b different from the width b_s of the conveying streambed, are becoming unworkably involved. The following approximate formulae, however, may be used if the height of the jump is not too great, say less than half the depth:

Let h_1 be the lower level ahead and h_2 the higher level in the rear of a jump, which travels up a river in the negative sense of x. Let H_1, Q_1, v_1, c_{01}, Z_{01} etc. be associated with h_1 and H_2 etc. with h_2. Then the velocity c of the jump is approximated by

$$(521) \qquad c = \tfrac{1}{2}(c_{01} - v_1) + \tfrac{1}{2}(c_{02} - v_2),$$

and between the states of motion separated by the jump, a relation approximated by

$$(522) \qquad H_2 - H_1 + (\tfrac{1}{2}Z_{01} + \tfrac{1}{2}Z_{02})\,(Q_2 - Q_1) = 0$$

exists.

The jump moves faster than the concurrent wave points ahead, but slower than the concurrent wave points in the rear. So both kinds of wave points meet the jump and merge into it. Considered by an observer moving with the jump, the flow in the low level region appears as supercritical and that in the high level region as subcritical.

The jump is progressive, i.e. moving toward the low level region, if $c > 0$. Such are the jumps in tidal regions as a rule. When $c < 0$, the jump would be regressive. We have to deal with the well-known stationary jump if $c = 0$.

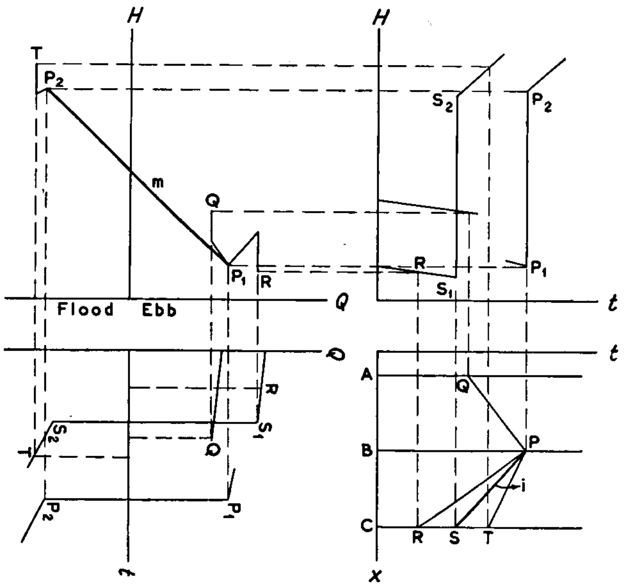

Fig. 11. Constructions to account for a jump (bore) in a tide.

In order to demonstrate how a jump is accounted for in a tidal computation by characteristics, let fig. 11 represent part of such a computation. Suppose the jump at the instant of passing by the place C (*tx*-points S) has been determined. This is represented by the discontinuities S_1S_2 in the *Ht* -diagram and the *Qt* -diagram. We set out to determine the jump as it passes by the place B. To that purpose we consider three wave points, all meeting the jump in B. They are: a concurrent wave point running ahead (itinerary RP), a concurrent wave point coming in the rear (itinerary TP), and a wave point encountering the jump (itinerary QP).

First we compute the velocity *c* of the jump according to (521), using estimations where necessary, and we construct the *tx* -point P by drawing the line i through S by the slope *c* to 1.

Next the state of motion (*HQ*-point P_1) just in front of the jump when it reaches B, is determined by considering the concurrent wave point ahead and the wave point coming from A. Then, according to (522) we draw a line m through the *HQ*-point P_1 by the slope $\frac{1}{2}(Z_{01} + Z_{02})$ to 1. This line together with the construction associated with the wave point following the jump, defines the *HQ* -point P_2 representing the state of motion just in the rear of the jump when it reaches B.

6. CONCLUSIONS

6, 1. Comparative appreciation of computation methods.

As the rather great variety of the computation methods presented in this paper may appear somewhat bewildering, we shall now endeavour to give a comparitive appreciation.

It is clear that, if there were a simply formulated exact solution of the mathematical equations of the tidal motion, there would be no need for any other solution. The tidal problems however are so intricate that such a solution is not possible and so we must accept the existence of various methods approaching the problem in different ways.

From the point of view of applicability we have to distinguish between approximate and exact methods. By an exact method we understand a method by which the solution of the mathematical problem can be determined to any desired degree of accuracy. This means that the accuracy which can be obtained practically is limited only by the accuracy of the observational data from which the computation starts.

The improvement of the accuracy generally goes at the cost of more labour. It depends on the purpose pursued how far one should go. For an explorative investigation approximate computations generally will do. A more detailed investigation requires greater accuracy and then it depends for a good deal on the nature of the problem which method is the most appropriate.

For the execution of the computation it makes a great difference whether trial and error procedures have been accepted in the method, or the computation proceeds straight forward. Whereas a straight forward procedure can be entrusted to a relatively unexperienced computer, a trial and error procedure can only be efficient in the hands of a computer with great experience.

We shall now first discuss the methods more in detail and then conclude by considering the use of computing machines.

Harmonic methods. The single-harmonic method in many cases reproduces very satisfactorily the fundamental of a periodic tide. The linearization of the resistance which has to be based on an estimation of the discharges, can be improved by successive approximations. An experienced computer often can make a fair estimate at once. Further the method is a straight forward procedure demanding relatively little labour. It is very appropriate for rapid exploration.

The sine approximation of a tide is not always acceptable. The tidal flow in particular may deviate appreciably from the sine trend. Then one may adopt a procedure of successive approximations, such that every approximation is extended to one more higher harmonic than the preceding approximation. The first steps in this procedure generally improve the accuracy, but the further steps give lesser improvements although the necessary labour increases. It is even dubious if the whole procedure is convergent at all. Practically it is usually not economic to go beyond the second harmonic. This double-harmonic method is still approximate.

The application of a harmonic method to a complicated network of channels necessitates a special analysis to deal with the connections between the channels, which may demand an appreciable amount of labour in excess to the computations of the separate channels. The network analysis is a straight forward procedure (cf [28] Ch 4, sect 3).

The harmonic methods are particularly well suited to deal with periodic tides. This restriction is not necessary in principle. For firstly we may treat nonperiodic functions by Fourier integrals, but the practical difficulties then encountered are in fact prohibitive. Secondly it has been endeavoured to treat nonperiodic motions, in particular storm tides by suitable approximate functions which are readily integrable. Lorentz [12] tried to work on the assumption of periodically occurring storm surges, Mazure [17] put forward the approximation by an exponentially exploding sine function, and we might also try sums of real exponential functions. All these artifices can not lead but to approximate methods because an improvement of the accuracy along these lines requires excessively much labour. For accurate computations of nonperiodic motions we may therefore exclude the harmonic methods.

Direct methods. Among the methods of quad-differences, power series and iteration, the latter is the most refined. It lends itself very well for the numerical analysis of observed tidal motions, which forms the indispensible check of the schematization (cf 2, 3). Therefore the iterative method is in particular efficient in estuaries which are hard to schematize.

Prediction by one of the above methods requires the simultaneous solution of a number of nonlinear equations. In systems of a few sections this can still be done in a straight forward manner. In greater systems one must accept a trial and error procedure. In networks of simple structure this is still feasible, but in complicated networks the trial and error labour rapidly becomes prohibitive.

The method of cross-differences is substantially a straight forward procedure. In Holsters' original form it is an approximate method, reproducing more details than a single-harmonic method but also requiring more labour. For explorative computations it may be appropriate. As soon as the method has to be refined, the labour involved increases rather rapidly.

Characteristic methods. The most profitable simplification in this type of method, viz the neglect of the resistance, is seldom admissable in tidal problems. Computing the resistance demandes rather much labour. Therefore the characteristic methods lend themselves not so much for explorative as for detailed investigations.

In comparison to the iterative method, an exact computation along characteristics usually demands more labour when the tidal motion considered is an observed tide or not too much deviating from such tides, and the tidal system is not a too complicated network. One of the advantages of the characteristic method however, lies in the straight forward procedure which enables us to predict tidal motions in complicated networks with an amount of labour roughly proportional to the extent of the system.

Characteristic methods are particularly well suited to deal with waves of finite extent such as produced by locks, dam failures, etc.

Computing machines. The great amount of computing labour necessary to obtain accurate results on a tidal problem, asks for considering the use of a computing machine, in particular of a rapid electronic computer. The efficiency of such a machine depends largely on the possibility of arranging the computation in the form of a pro gramme according to which the machine can work on uninterruptedly. Tidal computations however lend themselves to this only partly.

According to Dutch experience a considerable part of the labour of a tidal computation has to be devoted to finding the most appropriate schematization of the tidal system. Here much depends on the judgment of the computer and for that reason a programme can not be well given. When a schooled computing team has been formed to deal with this preliminary work, it can also deal with the proper predicting computations. Hence the use of a programme computing machine has only come into consideration in the Netherlands since very recently the task of the tidal hydraulicians was increased considerably.

The use of a computing machine can hardly be expected to be economic for approximate computations. To set up a programme, a straight forward procedure is moreover requisite. Hence the methods of characteristics and cross-differences are the most promising in this way of approach.

6, 2. Comparative discussion of computations and model research.

Model research and computation methods both pretend to yield solutions of tidal problems. A comparison therefore should not be omitted.

There are mainly two types of models: hydraulic models and electric analogues, which we describe briefly:

Hydraulic model. Geometrically true reduced scale models of extensive tidal systems should be made very large in order to observe certain limits to the scale reduction, firstly because the Reynolds' number in the model should be sufficiently great and secondly in view of the accuracy of measuring the vertical tide. As both conditions apply substantially to the vertical scale, models have been adopted in which the geometrical similarity is no longer strictly observed (distorted models). Such models are necessarily more or less schematized in a degree depending on the rate of distortion, and a true representation of the local flow patterns is no longer assured. Also the distribution of velocity in a cross-section is affected, but it can be demonstrated that the total discharges are truly represented provided the resistance is increased in an appropriate manner (exaggerated roughness). In fact this ought to be checked section by section.

A hydraulic model visualizes the water motion very clearly and directly which may be a great help for the engineer. As a disadvantage of hydraulic models may be noted that it is often difficult (partly because of the exaggerated roughness) to measure the variable discharges.

Electric model. On the basis of the analogy of electric and hydraulic systems [35] it is in principle possible to make an electric analogue of a tidal system [26]. Between the principle and the practical execution there is a long way. The model is made section-wise so that each model section represents a corresponding channel section in such a way that the total storage, inertance and resistance are truly represented. The internal mechanism in the model section offers no analogy with the channel section. The electric model therefore only represents truly the total discharges, which means that it is a schematic model. The electric currents analogous to these discharges can be measured as accurately in principle as the potentials representing the heads.

An electric model does not visualize the tidal motion as a hydraulic model does. It offers great possibilities however to produce very rapidly visual records of a great variety of tidal diagrams.

The choice between computations and models may be governed by the following considerations:

1. *Typical advantages.* Models offer rapid visualization possibilities. On the other hand computations enable us to penetrate more deeply into the physical mechanism of the motion, thus improving the insight. This is in particular true for computations with graphs and slide rules, and in a much lesser degree for computations with programme computing machines.

It depends partly on the nature of a tidal problem, whether a model or computations will be most appropriate, and often both means may profitably be applied in cooperation.

2. *Availibility.* Not always the facilities for both computing and model research are at hand. Model research requires a well equipped laboratory and computations require trained computers.

3. *Accuracy.* The accuracy of both models and computations is limited by the errors of many preliminary data, such as the Chézy coefficient. Also computations as well as distorted hydraulic and electric models are to a certain extent dependent on schematization. A generally valid comparison as to the degree of the accuracy between the three seems hardly possible as the means by which it can be tried to reduce the deviations from nature are not identical.

In hydraulic models both the effects of schematization and the measuring errors can be decreased by the use of larger models. Here the economy becomes a very important argument (v. below).

In an electric model the accuracy depends on the schematization and partly on the veracity of the representation of hydraulic properties (such as the quadratic resistance) by electric elements. Whereas the former can be improved rather easily by increasing the number of sections by which the schematization is set up, the latter can only be improved as far as the electric technique admits.

The accuracy of a computation can be improved by refining the schematization and, if we consider exact methods, by proceeding to a further approximation.

4. *Economy*. There is a rather great difference between models and computations in the ratio of variable to fixed costs. The building of a model is a rather expensive affair. Once the model built however, it is relatively easy to deal with a great variety of problems in the same tidal system. The preparations for a computation (schematization, checking, etc.) likewise take much labour, although relatively less than in case of a model, whereas the investigation of a number of problems takes relatively more labour than in a model.

In the Netherlands both types of models as well as several of the methods of computation treated in the preceding chapters, play a useful part in the solution of the manifold problems with which the tides confront us.

Postscript

When the preceding paper was conceived, the possibilities offered by modern computation techniques, digital or analogous, had only been explored provisionally. In the four years elapsed since, considerable progress has been made.

Digital computing is now being done by a method based on a quad network, in which the integrations are performed along characteristics, which are fit into the quad scheme by interpolations. The quad scheme is preferred to a staggered (cross) scheme, because in the staggered net the administrative part of the programme governing the procession through the net, becomes much more complicated than in case of a quad net. The integration along the characteristics has the advantage of easy adaptation to any type of terminal conditions.

For dealing with the mass of problems in the Delta region, a new electric analogue computer is now being constructed. This analogue will largely supersede the electronic analogue which has been in operation since 1953. The new computer is capable of simulating the equations (201) and (203) for arbitrary cross-sections. All types of boundary conditions can be introduced.

The accuracy of the analogue is checked by means of digital computations.

november 1958

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REPORT ON HYDROSTATIC LEVELLING ACROSS THE WESTERSCHELDE

A. WAALEWIJN [1]

SUMMARY

In 1952 the Surveying Department of "Rijkswaterstaat" carried out a hydrostatic levelling across the "Westerschelde", using an underwater gas-pipe of 10 cm diameter and a length of more than 4 km. Great difficulties had to be overcome in filling the pipe completely, because a large air-bubble formed in the middle of the nearly W-shaped longitudinal profile of the pipe. Finally this air-bubble could be removed by fitting taps and moving the water mass by means of a big pump. The observations were carried out from dec. 16 to dec. 24, 1952. The water level in the pipe ends was measured by means of automatic water-gauges recording on scale 1:1.
Microbarometers were used in measuring the atmospheric pressure at both ends of the pipe. It appeared, however, that these observations were useless; therefore the correction for difference in atmospheric pressure had to be obtained from data of the adjacent meteorological stations.
Bottom-temperatures in the Westerschelde could be measured only once. The results of this measurement were such that no corrections for difference in density were necessary.
The standard deviation of a single observation of the (corrected) height difference was 1.1 mm. The average result of the hydrostatic levelling has a standard deviation of 0.2 mm. By this levelling a 120 km long circuit from Zuid-Beveland via Woensdrecht and Antwerp to Zeeuws-Vlaanderen was closed. The misclosure of this circuit was 0.3 mm. Further the report gives the results of some experiments: The swinging of the water mass after disturbance of balance, the influence of low tide and high tide on the capacity of the pipe and a non-stop series of microbarometer-readings.

SOMMAIRE

En 1952 le Service Topométrique du «Rijkswaterstaat» a exécuté un nivellement hydrostatique à travers le Westerschelde, en utilisant un déversoir à gaz avec une longueur de plus de 4 km et un diamètre de 10 cm.
Le remplissage du déversoir avec de l'eau offrait de grandes difficultés, parce que le profil en long du tube était en forme de W, de sorte qu'une grande bulle d'air se forma au milieu du tube. Cette bulle d'air fut expulsée par le montage d'un robinet et le mouvement de la masse d'eau au moyen d'une pompe.
La différence d'altitude des deux stations séparés par le Westerschelde fut observée du 16 décembre jusqu'au 24 décembre 1952. Le niveau d'eau dans les extrémités du tube fut mesuré par un marégraph enrégistrant à l'échelle 1:1.
La pression atmosphérique des deux stations fut mesurée avec des microbaromètres.
Ces observations se révélèrent inutilisables; c'est pourquoi la correction pour la pression atmosphérique fut calculée à l'aide des données des stations météorologiques circonvoisins.
Les températures au fond du Westerschelde ne pouraient être mesurées qu'une seule fois. Les résultats de ces observations étaient tels qu'aucune correction pour des différences de densité paraissait nécessaire.
L'erreur moyenne d'une seule observation de la différence d'altitude, corrigée pour la pression atmosphérique se trouve 1.1 mm. L'erreur moyenne de la moyenne de toutes les observations est 0.2 mm. Ce nivellement hydrostatique ferme une polygone de 120 km avec une erreur de fermeture de 0.3 mm. Le rapport donne aussi les résultats de quelques expériments: l'oscillation de la masse d'eau après perturbation de l'équilibre; une série d'observations simultanées des microbaromètres avec des intervalles de 1 minute; l'influence des marées du Westerschelde sur la capacité du tube.

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1. INTRODUCTION

1, 1. Motive of the report.

There are three means that can be employed to increase the accuracy with which the N.A.P. (Amsterdam ordnance datum) is fixed at any given point in the Netherlands: the application of specific methods of measuring by which various errors can be avoided or eliminated, observations made by well-qualified and experienced observers with very good instruments, and the measuring of closed circuits so that a well coordinated network of measurements is achieved.

Fig. 1 presents the net of the second Dutch geodetic levelling, showing that in the south-western part due to its deltaic character no circuits have been measured. Precisely in this district the coordination of the zeropoints of the tide-gauges is very important because the local movements of the water must be known here exactly for the preparation and execution of the Delta Works. Especially the accuracy of the tidal calculations before and during the enclosures of the estuaries depends largely upon very accurate heights of the zero-point of gauges.

The lack of direct connections across the estuaries, makes the mutual accuracy of the heights on either side of each stream less than it could be if those estuaries could be crossed. The measurements across the estuaries are called „river-crossings".

The linking-up of the levellings on the Wadden-Islands in the nothern part of the country with those on the mainland presents similar problems; the difficulties to be surmounted are even greater. But no closings of sea-entrances are planned here for the moment.

1, 2. Various river-crossings.

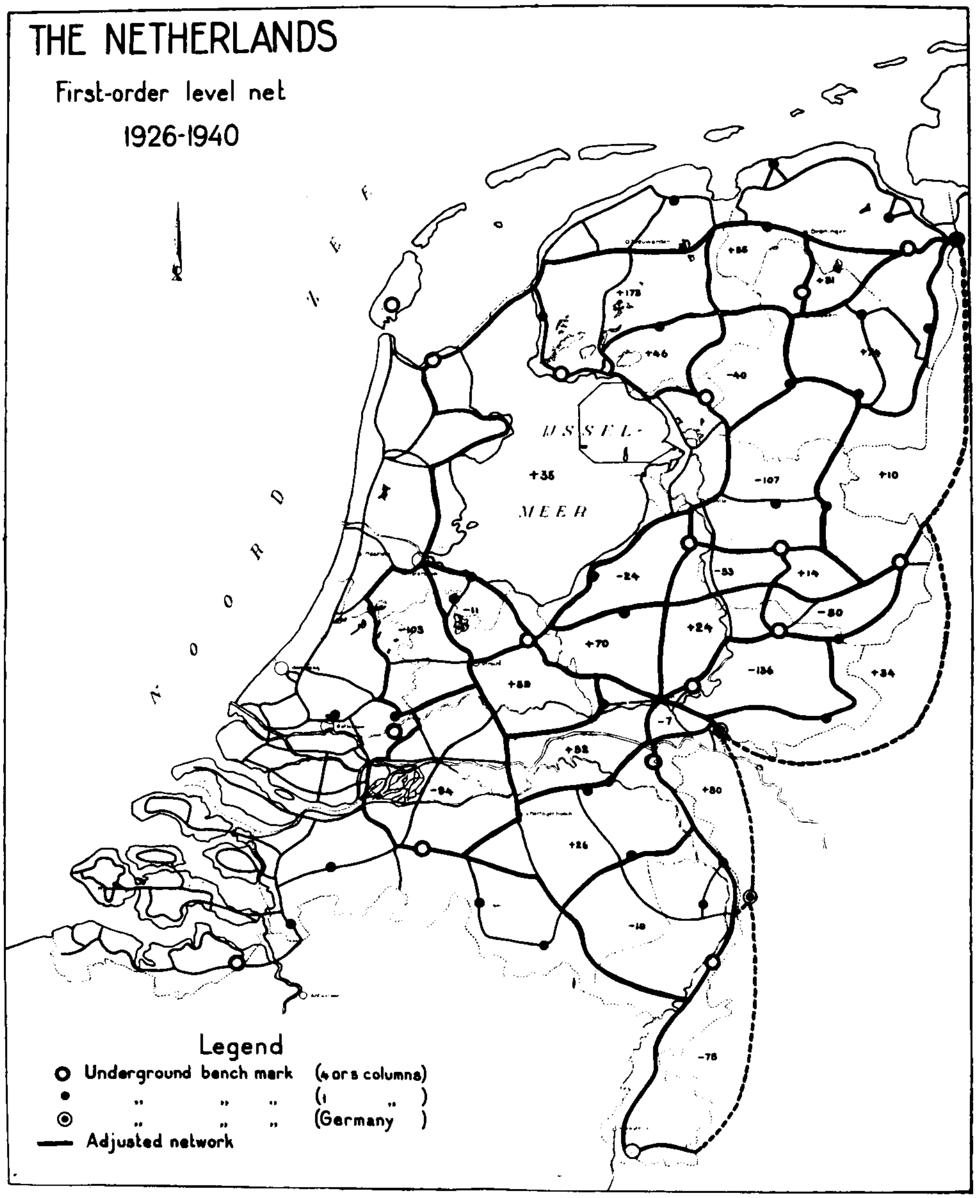
The measurement of river-crossings is the most difficult task in the execution of precise levelling. The water surface in the estuaries is not a level surface, because of tidal movements, windinfluences, waves and differences in density, and therefore cannot be used.

In the Netherlands large rivers and even estuaries were crossed with the normal optical method of reciprocal levelling, even though it be with special precautions.

The most important errors which occur with the optical method are due to:

the curvature of the earth,
atmospheric refraction,
maladjustment of the instrument.

Atmospheric refraction is above all a source of errors that are difficult to eliminate whereas also a local discrepancy in the curvature of the geoïd may cause a systematic error which cannot be eliminated.



1 : 2.000.000

Fig. 1

Some examples of optical river-crossings in the Netherlands after 1920 are,

1922	Marsdiep	2 km	}	linking up Frisian islands
,,	Eyerlandse gat	2 km		
,,	Vlie	6 km		
1930	Volkerak - Hellegat	2 km	}	in the Southwest (Zeeland)
1952	Volkerak	1.5 km		
,,	Haringvliet	1 km		

Great difficulties were met with especially during the crossing of the Vlie by W. Schermerhorn, who says in his report that the measurements of such large crossings (6 km.) is not to be recommended.

Even in the case of the relatively small river-crossings measured in 1952 there appeared to be so much uncertainty that, for the present, no optical crossings larger than 2 km. will be performed. Since the widths of the estuaries in Zeeland vary from 4.5 to 10 km, it is obvious that optical means cannot be applied there.

In Denmark the same problems occur. At the end of the 19th century optical crossings were made over the Great and Little Belts with a maximum distance of 9 km. (described by General Zachariae). However, on the occasion of the renewal of the Danish geodetic levelling in 1938 the optical crossings were considered as being too uncertain, so that it was decided to use another system, namely the hydrostatic method.

A very big tube-level was used: a leaden hose completely filled with water was laid out in the Belt between the two banks. According to the law of communicating vessels the water surfaces in the hose on both banks will lie in the same level surface provided that

a. there is no air in the tube which divides the water into two, or more, non-communicating parts,
b. a correction is made for differences in air-pressure above both surfaces,
c. a correction is made for unequal density by reason of differences in temperature in the water in the rising sections of the tube.

A similar process of measuring was carried out over the Øresund (1939). These hydrostatic levellings form a link in the international chain of levellings which ring the Baltic.

1, 3. Underwater gaspipes in the Westerschelde.

Inspired by these Danish measurements consideration was given to the idea that the method might offer a solution for creating a number of circuits in the south-western part of the Netherlands. In 1951, for the benefit of the supply of gas to South-Beveland and Walcheren, three steel pipes were laid through the Westerschelde from the dyke near the hamlet of De Griete (Zeeland-Flanders) to the dyke near Baarland (S. Beveland).

The Directorate of Energy gave permission to use these pipes for a hydrostatic levelling over the Westerschelde before they were put into normal service. By means of this levelling the circuit S. Beveland-Woensdrecht-Antwerp-Zeeland Flanders would be closed, while at the same time experience with this method of levelling could be gained.

The length of the pipes was 4200 metres, the internal diameter 96 mm., making the capacity of each pipe about 30.400 litres. To avoid affecting its inner lining only fresh water could be used for filling the pipe, and as no local water-mains were available the water required had to be brought by boat from Terneuzen harbour.

A sand-bank, called Rug van Baarland, lies almost in the middle of the Westerschelde making the shape of the cross section of this part of the estuary something like the letter W (fig. 2). Because of this shape an air-bubble would occur on the sand-bank when filling the gas-main with water.

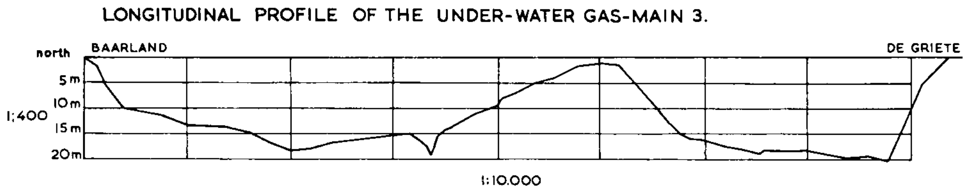


Fig. 2 Longitudinal section of Westerschelde at gas-main number 3.

To remove such an air-bubble two different methods seemed to be possible:

using a large pump to cause the water in the pipe to be in rapid motion so that the air-bubble, if not altogether swept away, could yet be broken up and thus removed bit by bit,

placing a tap on the highest point of the pipe on the sand-bank through which the air could escape.

Preference was given to the first method because it was of a less drastic nature; only on the failure of this method would taps be used.

In order to limit the use of fresh water it was decided to handle two pipes at one and the same time by fastening them together and then, by means of the pump, causing a circulation of water. Seeing that one of the pipes, namely number 1, is separated by almost 400 metres from the other two, this pipe was considered as being the least suitable and was not made use of; moreover if this pipe were used it would be necessary to erect an extra observation post with personnel for the hydrostatic levelling. Thus only pipes number 2 and 3 were prepared in this way for the work on hand.

2. THE FILLING AND FREEING FROM AIR

2, 1. The filling of the pipes; estimation of air-bubble.

At the end of September 1952 the preparatory work for the hydrostatic levelling could start by filling the underwater gas-mains by the tanker-boat from Terneuzen.

As the time during which this boat could be anchored near the shore at Baarland was limited, being only about high-tide, and, as the pressure given by the boat's pump was only small, success in filling a pipe at a single attempt was not achieved. The tanker-boat had to return twice before both pipes were filled. There was, then, a large amount of air in the pipes demonstrated by the fact that on removing the pressure after having pumped the pipes full, much water flowed back out of them. Apparently the trapped air expanded after the pressure was released, thus forcing out a quantity of water. It appeared possible, if certain points were taken for granted, to determine the volume of air in the pipe. The addition, on one side, of a known quantity of water would cause a rise of water at the ends of the pipe from which deductions could be made.

The points taken for granted were:
Only **one** air-bubble was present in the pipe.
This bubble was situated on the sand-bank.
The slope of the pipe was equal at both the extremities of the air-bubble.

According to Boyle's Law VP = constant, while
V = volume of bubble, and P = pressure in bubble.

Then $VdP + PdV = O$, in which

dP = increase of pressure inside the bubble. This increase can be estimated from the rise in the water surfaces after the addition of a known quantity of water,

P = original pressure inside the bubble; to be calculated on the assumption that it was situated above the sand-bank whose height in reference to the ends of the pipes was more or less known.

dV = change in volume of the bubble; to be measured as the difference between the known quantity of added water and the measured rises on both ends.

V = volume of the bubble; to be calculated from the above formular using these data.

Only dV could be measured accurately; the estimation of P was fairly correct, while the value of dP was dependent on the rises of the water surfaces at the ends of the pipes and those of the water surfaces next to the air-bubble where, however, the slopes of the pipe were unknown. From two of these tests these unknown slopes could be calculated, however. It appears on looking back, that the assumption that both slopes are alike had been fairly correct.

2, 2. The pumping-through of the gaspipes.

After the gas-mains were filled and the connection between the two pipes completed the pumping-through could start. For this purpose the Fire Service in Goes co-operated by putting a pump at our disposal. By means of this pump, set up on the dyke near Baarland, success in obtaining a water circulation in both pipes was achieved; circulatary pumping was done throughout a whole afternoon. At the finish it was clear that still much air was present in both pipes.

Seeing that the pressure could only be raised to 4 atmospheres on account of the limited water quantity, and seeing, moreover, that the speed of the circulating water was unfavourably influenced by the length of the two pipes coupled together, still another method was tried.

In this case only one pipe (the number 3) was pumped through, the tanker-boat supplying the water that was forced into the pipe by means of the Fire Service's pump. This water could then flow into the Westerschelde at the other end of the pipe. The speed of the current achieved was now considerably greater, namely 300 litres per min. as against 200 litres per min. at the first attempt.

At the end of this pumping process, however, no decrease in the volume of air in the pipe could be reported. A number of experiments done to ascertain the volume of air showed that there was still an air-bubble in pipe 3 with a volume of 1175 litres (estimated standard-deviation $m_v = 30$ litres).

This volume meant an air-bubble of 163 metres length. The slopes of the pipe on each side of the bubble calculated from the experiments were: $\text{tg}\ a_1 = 0.053$ $\text{tg}\ a_2 = 0.058$.

Because of the very slight success achieved by this pumping it was decided to remove the air-bubbles by fitting taps on the pipes on the sand-bank.

This was done during low-water at spring-tide in the beginning of November. Much air escaped from the taps, but a new experiment demonstrated that not nearly all the air had been expelled:

for pipe 3 we found $V = 535$ litres; $m_v = 30$ litres and $\text{tg}\ a = 0.03$.

This decrease, coupled with the presence of taps near where the remaining air-bubbles must be, gave occasion for trying once more to expel the air by circulatory pumping while an observer was on the sand-bank whose duty it was to let any air-bubbles, which might pass, escape from the taps. This time the process of pumping was more successful as the following measurements showed:

pipe 3 : $V = 390$ litres with $m_v = 12$ litres and $\text{tg}\ a = 0.005$
,, 2 : $V = 390$,, ,, $m_v = 12$,, ,, $\text{tg}\ a = 0.03$

Success in making the pipes air-free was attained after having pumped 3 times more.

Unfortunately on the last day it became evident that pipe 2 had a leak, probably caused by a ship's anchor, so that for the execution of the hydrostatic levelling only pipe 3 remained. A unique opportunity of executing two hydrostatic levellings simultaneously at the same place was lost.

3. THE MEASURING OF THE HYDROSTATIC LEVELS

3, 1. The establishment of observation-posts.

Wooden observation huts were placed over the ends of the pipes 2 and 3 on the dykes near De Griete and Baarland; a heavy wooden partition served as a means of fixing the various instruments.

Stand-pipes with a diameter of 20 cms. were placed on the ends of pipe 3. Automatic gauges were used to ascertain the heights of the water-surfaces in these stand-pipes. A float, whose vertical movement was changed over to a circular movement

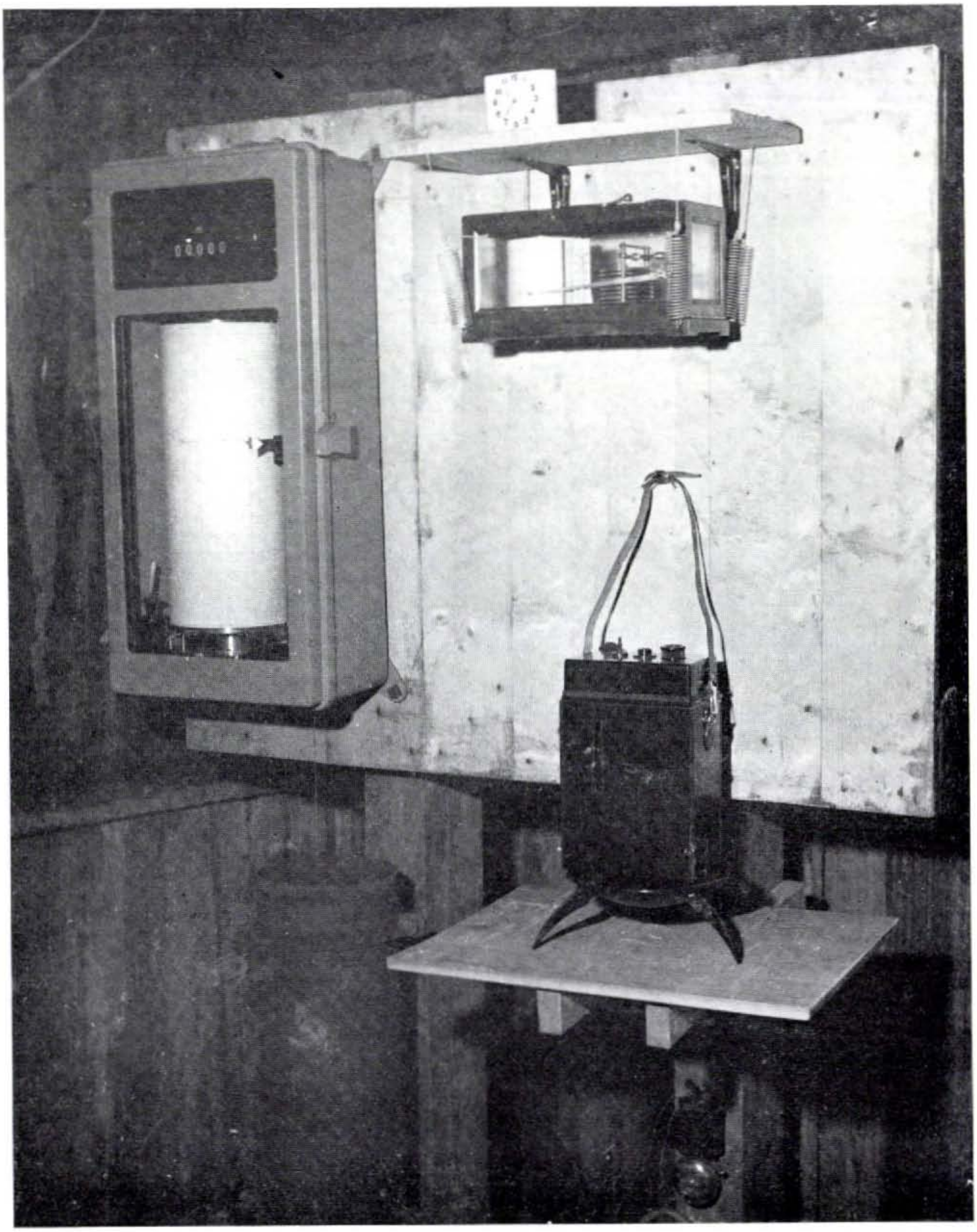


Fig. 3

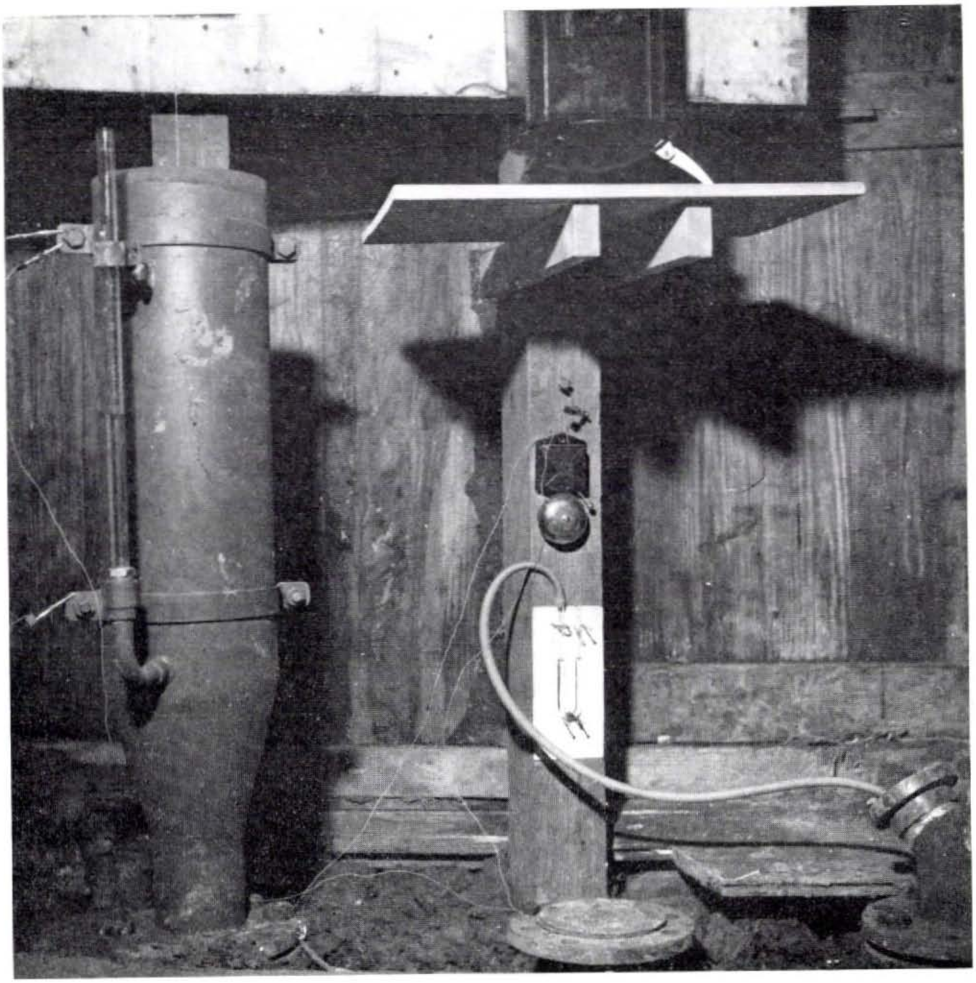


Fig. 4

round an axis by means of a wire, could move in the stand-pipe. This axis caused a pen-nib to move vertically along the surface of a drum bearing a sheet of paper and revolving on a vertical axis. The drum revolved, by means of clockwork, once in $7\frac{1}{2}$ days so that the nib drew a graph of the water-level in relation to the time taken. The recording of the water-level was done to true scale; the time scale being 3 mm = 1 hour.

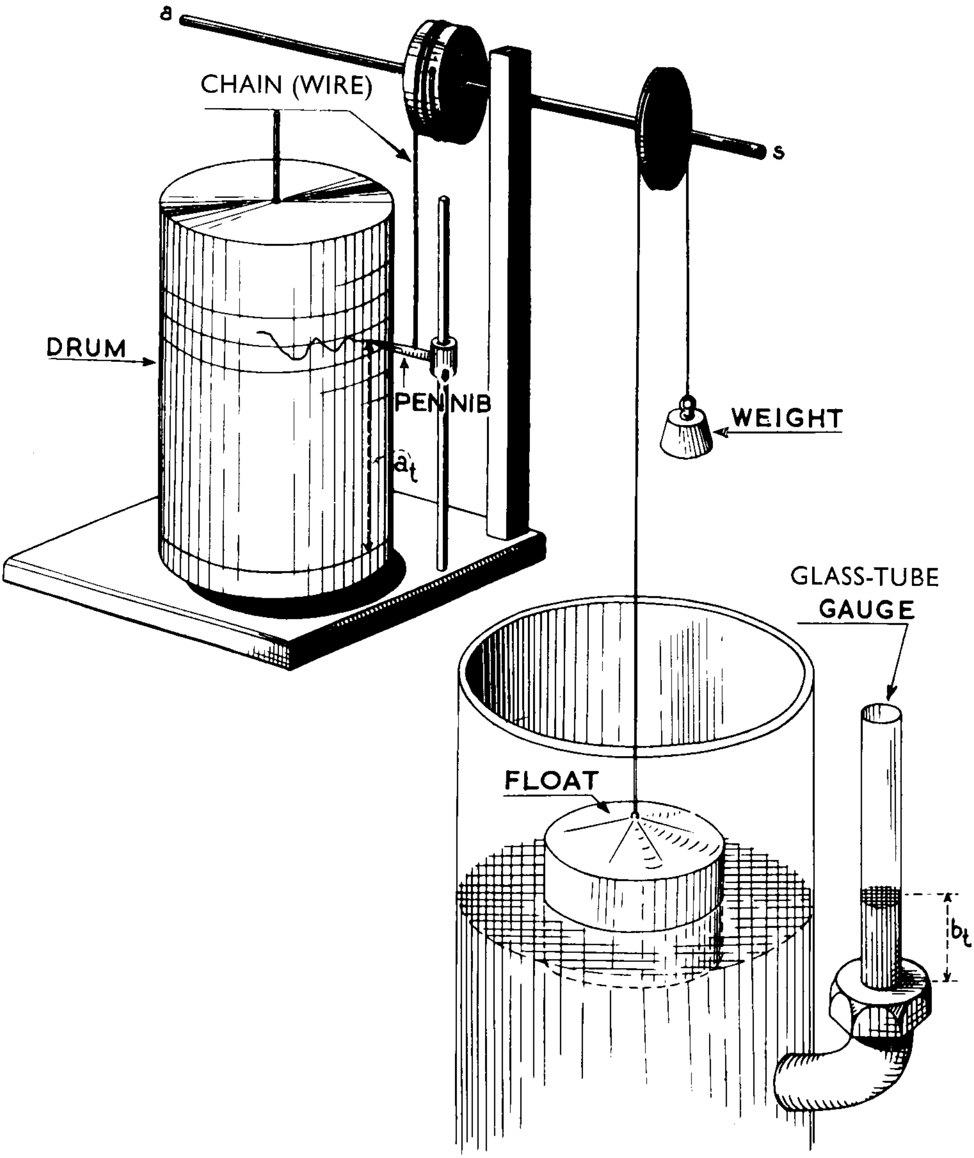


Fig. 5

As an automatic gauge has always to be verified against a normal one, such normal glass-tube gauges had been attached to the stand-pipes.

A barograph was hung at each observation-post for measuring the atmospheric pressures. These instruments being not precise were only intended for ascertaining the atmospheric pressure between the moments when more precise instruments were being read. A micro-barometer (of the make Askania) was also placed on each bank, which makes the reading of the atmospheric pressure possible down to 0.01 mm. of mercury; it is designed for barometric measurements of heights. (Fig. 3 and 4)

An electric bell had been fixed to connect the two observation posts on either side of the estuary so that signals could be transmitted. The current needed was provided by an accumulator.

Two so-called "handy-talkies" served to carry over reports. The instruments had also done good service during the preparations for the work.

3, 2. The measuring.

At each post three observers were present who, from Dec. 16th to Dec. 24th, inclusive, made the readings required. These readings were:

a. *Readings of atmospheric pressure.* After a signal given on the bell the micro-barometers were read every hour simultaneously. Moreover, on some afternoons these microbarometers from both banks were brought together and compared in order that any difference in readings could be known as accurately as possible. However, seeing that these comparisons caused too much delay, there being no service-boat at our disposal at the moment, a stop was put to these comparisons. Another reason for putting a stop to it was that a non-constant difference in reading would be difficult to take into account, whereas a constant difference could always be taken into account later.

On the 22nd of December the microbarometers on both banks were read minute by minute during several hours so that a practically continuous picture of the course of the barometric heights was obtained on both sides.

b. *Observations on the glass-tube.* The water-levels at the gauges were read hourly with a pocket-rule held against the copper nipple of the glass-tube. Owing to a misunderstanding at De Griete the top level of the meniscus was repeatedly measured while at Baarland the bottom level was read. Because of this a systematic error came into being, the size of which was measured later and stated to be 2.6 mm.

Because the surface of the water in the stand-pipes sometimes oscillated very violently — with a maximum amplitude of about 6 cms — the readings on the gauges were not always reliable. Now and then the clockworks in the drum of the gauges were checked; by moving the float somewhat, a vertical line on the water-level graph was made. The exact time of that line was then written in. The readings of atmospheric pressure and gauges were as a general rule made between 9 a.m. and 6 p.m. with a supplementary reading made later in the evening, between 10 p.m. and mid-night.

c. *Linking up with the first-order levelling.* Some few times levelling was done on both banks from the zeropoint of the glass-tube — in this case the copper nipple — to a bench-mark of the first-order levelling system.

The stability of this bench-mark was further checked by measuring to one or more adjacent bench-marks.

d. *Measuring the temperature on the bottom of the Westerschelde.* The original plan to measure the sea-water temperatures on the bottom alongside the underwater gas-mains each day could not be realized as on the first days the weather was extraordinarily rough, whereas later the motor-boat "Westerschelde" equipped for the purpose of tidal observations, was not available owing to activities elsewhere.

It was only on the 23rd December, under almost ideal weather conditions, that measurements of temperature could be made. During the turn of the tide a reversing thermometer was sunk alongside the gas-mains in order to measure the temperature of the water at the bottom. As the thermometer requires about 10 mins. to register the temperature of its surroundings, only about 15 mins. (sailing time included) were available for the measurement at each point. Considering that only for about 2 hours the speed of the current is small enough for the boat to lie reasonably still, only 8 points per turn of the tide could be measured.

The reversing thermometer is a thermometer which enables us to fix the temperature attained at a certain moment by making the thermometer turn over at that particular moment. By so doing a quantity of mercury is separated which is decisive for the temperature at that precise moment. The temperatures showed practically no differences. In order to ascertain whether this was normal, temperatures were measured several times more after the actual periods of observation. In this way the results found previously were confirmed.

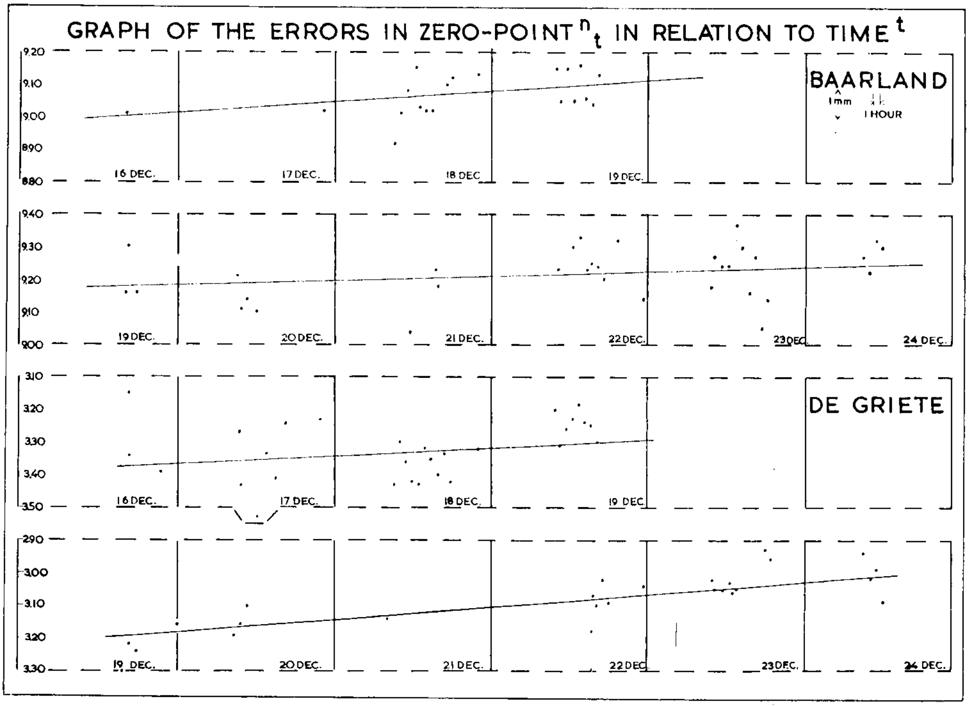


Fig. 6

TABLE I

A comparison of the heights in relation to the zero-line measured on the graph and those read on the glass-tubes expressed in cms.

Baarland					De Griete				
	t	a_t	b_t	n_t		t	a_t	b_t	n_t
Date 1952	Time	Graph measurements	Gauge measurements	Difference	Date 1952	Time	Graph measurements	Gauge measurements	Difference
Dec. 16	16.40	25.56	16.55	9.01	Dec. 16	16.50	14.15	17.30	– 3.15
						16.55	14.16	17.50	– 3.34
Dec. 17	22.15	25.42	16.40	9.02		21.30	14.11	17.60	– 3.49
Dec. 18	9.00	24.97	16.05	8.92	Dec. 17	9.45	14.23	17.60	– 3.27
	10.00	25.06	16.05	9.01		10.05	14.22	17.65	– 3.43
	11.00	25.08	16.00	9.08		12.40	14.12	17.65	– 3.53
	12.08	25.05	15.90	9.15		14.00	14.17	17.50	– 3.33
	13.00	25.18	16.15	9.03		15.30	13.99	17.40	– 3.41
	14.00	25.22	16.20	9.02		17.00	14.01	17.25	– 3.24
	15.05	25.22	16.20	9.02		22.05	14.07	17.30	– 3.23
	16.00	25.25							
	17.15	25.20	16.10	9.10	Dec. 18	9.00	14.17	17.60	– 3.43
	18.05	25.07	15.95	9.12		10.00	14.20	17.50	– 3.30
	22.03	25.03	15.90	9.13		11.00	14.24	17.60	– 3.36
						12.00	14.18	17.60	– 3.42
Dec. 19	9.25	25.05	15.90	9.15		13.00	14.17	17.60	– 3.43
	10.00	25.05	16.00	9.05		14.00	14.18	17.50	– 3.32
	11.00	25.05	15.90	9.15		15.00	14.15	17.50	– 3.35
	12.00	25.05	16.00	9.05		16.00	14.10	17.50	– 3.40
	13.00	25.06	15.90	9.16		17.00	14.07	17.40	– 3.33
	14.00	25.06	16.00	9.06		18.00	14.08	17.50	– 3.42
	15.00	25.14	16.10	9.04		22.00	14.08	17.40	– 3.32
	16.00	25.18	16.05	9.13	Dec. 19	9.15	14.00	17.20	– 3.20
						10.00	13.99	17.30	– 3.31
						11.00	14.04	17.30	– 3.26
	16.35	25.21	16.05	9.16		12.00	14.07	17.30	– 3.23
	17.04	25.20	15.90	9.30		13.00	14.12	17.30	– 3.18
	18.00	25.16	16.00	9.16		14.00	14.16	17.40	– 3.24
	24.00	25.11	15.85	9.26		15.00	14.15	17.40	– 3.25
						16.00	14.10	17.40	– 3.30
Dec. 20	9.15	25.01	15.80	9.21		16.00	14.10	17.40	– 3.30
	10.00	25.01	15.90	9.11		17.00	14.08	17.30	– 3.22
	11.00	25.04	15.90	9.14		18.00	14.06	17.30	– 3.24
	12.35	25.05	15.95	9.10		24.00	14.04	17.20	– 3.16
Dec. 21	11.40	24.88	15.85	9.03	Dec. 20	9.00	13.96	17.15	– 3.19
	15.52	24.88	15.65	9.23		10.00	13.94	17.10	– 3.16
	16.02	24.88	15.70	9.18		11.00	13.90	17.00	– 3.10
Dec. 22	9.43	24.68	15.45	9.23	Dec. 21	16.00	13.81	16.95	– 3.14
	12.00	24.65	15.35	9.30					
	12.52	24.63	15.30	9.33	Dec. 22	15.00	13.72	16.90	– 3.18
	14.20	24.63	15.40	9.23		15.39	13.73	16.80	– 3.07
	15.02	24.70	15.45	9.25		15.50	13.75	16.85	– 3.10
	16.05	24.69	15.45	9.24		16.00	13.75	16.85	– 3.10
	17.03	24.70	15.50	9.20		17.00	13.78	16.80	– 3.02
	18.40	24.82	15.50	9.32		18.00	13.76	16.85	– 3.09

	Baarland					De Griete			
	t	a_t	b_t	n_t		t	a_t	b_t	n_t
Date 1952	Time	Graph measurements	Gauge measurements	Difference	Date 1952	Time	Graph measurements	Gauge measurements	Difference
	23.37	24.64	15.50	9.14		23.38	13.71	16.75	- 3.04
Dec. 23	9.23	24.58	15.40	9.18	Dec. 23	9.22	13.68	16.70	- 3.02
	10.00	24.57	15.30	9.27		10.00	13.65	16.70	- 3.05
	11.00	24.54	15.30	9.24		11.00	13.65	16.70	- 3.05
	12.00	24.54	15.30	9.24		12.00	13.62	16.65	- 3.03
	13.05	24.52	15.15	9.37		12.30	13.59	16.65	- 3.06
	14.00	24.50	15.20	9.30		13.00	13.60	16.65	- 3.05
	15.00	24.51	15.35	9.16		14.00	13.59	16.65	- 3.06
						water added			
	16.00	24.52	15.25	9.27		17.30	16.82	19.75	- 2.93
	water added								
	17.00	27.65	18.60	9.05		18.00	16.79	19.75	- 2.96
	18.00	27.74	18.60	9.14					
Dec. 24	9.11	27.67	18.40	9.27	Dec. 24	9.00	16.76	19.70	- 2.94
	10.00	27.67	18.45	9.22		10.00	16.68	19.70	- 3.02
	11.00	27.67	18.35	9.32		11.00	16.66	19.65	- 2.99
	11.45	27.65	18.35	9.30		11.45	16.61	19.70	- 3.09

3, 3. Elaboration of the data.

a. In the first place with the help of a magnifying-glass, magnification about 6, the water-heights a_t were measured on the gauge-graphs at the times $_t$ at which the gauge-readings b_t were taken with reference to the nipple.

The difference in these values, $a_t - b_t$, is the zero-error $n\,t$ of the gauge-graph at the moment t. (fig. 5, table I, and fig. 6.)

The values n_t sometimes differed considerably while there appeared to be a tendency to a systematic change due to time. Because of this an adjusted value of the zero-point error N_t at any stated moment was always determined from a straight line which was drawn, as well as possible, through a series of points on the graph of n_t.

With this zero-point error N_t the adjusted water-level B_t with reference to the nipple on the gauge could be fixed at any given moment from the graph of the gauge:

$$B_t = a_t - N_t$$

The hydrostatically measured difference in height h_t of the nipples in Baarland and De Griete is then the difference of the values B_t taken simultaneously on both banks: nipple Baarland above nipple De Griete at moment t:

$$h_t = (B)_{t\ \text{de Gr}} - (B_t)_{\text{Brld}} \quad \text{subject to corrections (fig. 7)}$$

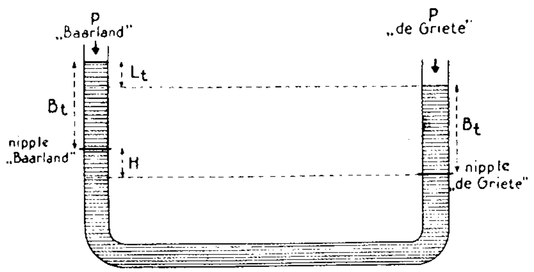


Fig. 7

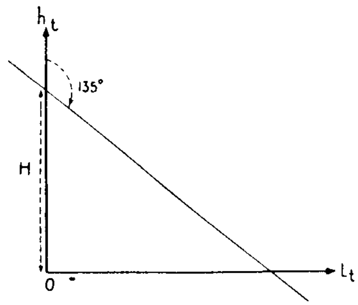


Fig. 8

b. Since, theoretically, a difference in atmospheric pressure L_t between the two banks influences the water-levels, h_t is subject to correction.

Therefore a table was made of the measured difference in atmospheric pressure l_t (Askania microbarometer) and also one of the differences in heigth h_t. If a connection exists between the two this must appear on a two-dimensional correlation chart in which each moment t is indicated by a point with co-ordinates l_t and h_t. (Fig. 9)

Theoretically:

$$L_t = (P_t)_{\text{Brld}} - (P_t)_{\text{De Gr.}} \quad \text{(expressed in mm of water)}$$

in which P_t = atmospheric pressure at moment t,

while (see fig. 7) the exact difference in the height of the nipple Baarland above the nipple De Griete is H then

$$H = (B_t)_{\text{de Gr.}} + L_t - (B_t)_{\text{Brld}} = h_t + L_t$$

Therefore the theoretical connection between h_t and L_t is a straight line making an angle of 135° with the y-axis (fig. 8).

An index-difference existed between the two microbarometers in use. If this difference is constant and equal to a certain amount C, then the theoretical relationship must be

$$H = h_t + l_t - C$$

This correlation graph, however, shows no indication of such a relation. On these grounds it was decided not to make a correction for these differences in atmospheric pressure l_t. The fallacy of the correction could be shown again in another way, namely by fixing the accuracy of the uncorrected differences in height and of these when corrected by (l_t — C), as follows:

standard deviation in uncorrected h_t = 1.28 mm.
,, ,, ,, corrected $h_t + l_t - C$ = 2.73 mm.

(when C is a fixed laboratory value)

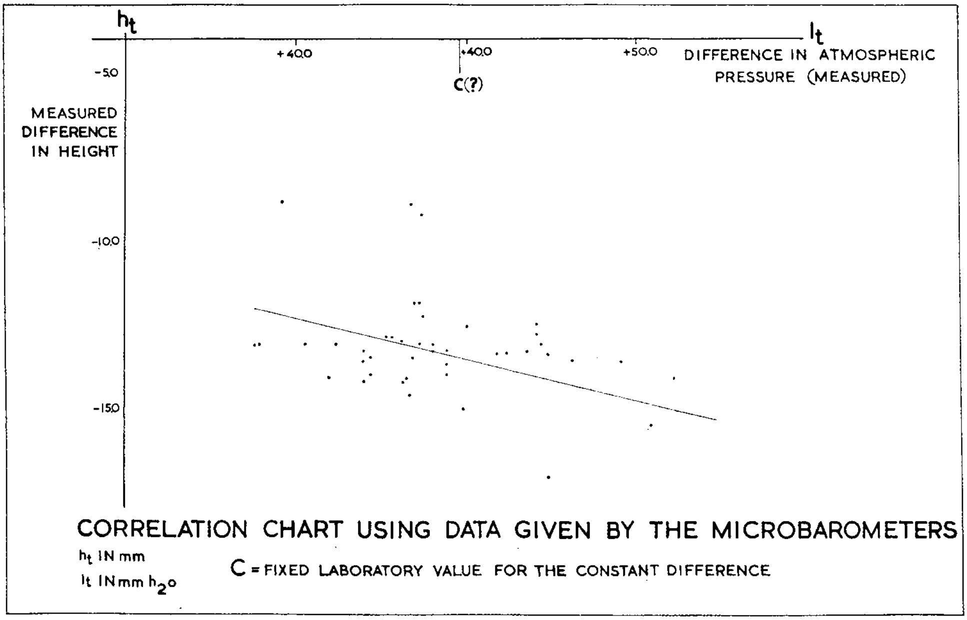


Fig. 9

c. It was feared that by not making a correction for atmospheric pressure a systematic error would result in the difference in heights eventually found, resulting from an asymetric build-up of the atmospheric pressure field throughout the whole period of observation.

Therefore the records of atmospheric pressure made by the neighbouring meteorological stations were investigated; these stations were the lightship Goeree, Hoek van Holland, Woensdrecht, Antwerp, Brussels, Wevelgem, Coxyde, Vlissingen and sometimes others. From these data isobars were drawn with interval 1 mb. The difference of atmospheric pressure

$$m_t = (P_t)_{\text{Brld}} - (P_t)_{\text{de Gr.}}$$

was fixed by means of the interpolation of these figures.

The observation times at the meteorological stations were 0, 3, 6, 9, 12, 15, 18 en 21 hours G.M.T.; the heights of the water B_t for these times could again be measured in the usual way from the gauge-graphs (see table II).

A two-dimensional chart for h_t and m_t indicates indeed a connection (fig. 10). A calculation of the regression lines gave namely the following result:

$$h_t = 1.06\ m_t - 1.40$$
$$m_t = -\ 0.38\ h_t - 0.60$$

TABLE II

Differences in height and atmospheric pressure as given by the meteorological data

Date	Time MET	Difference in height h_t	Difference in pressure m_t	Corrected difference in height H_t
		mm	0.1 mb	mm
Dec. 16	16	− 10.0	− 2.7	− 12.7
	19	− 12.2	− 4.0	− 16.2
	22	− 11.4	− 2.6	− 14.0
Dec. 17	1	− 11.4	− 1.3	− 12.7
	4	− 10.5	− 1.6	− 12.1
	7	− 11.0	− 1.4	− 12.4
	10	− 11.6	− 1.8	− 13.4
	13	− 8.8	− 2.0	− 10.8
	16	− 8.2	− 2.7	− 10.9
	19	− 10.3	− 3.0	− 13.3
	22	− 10.1	− 2.7	− 12.8
Dec. 18	1	− 10.1	− 2.0	− 12.1
	4	− 12.7	0.0	− 12.7
	7	− 12.1	− 0.5	− 12.6
	10	− 15.1	− 0.1	− 15.2
	13	− 13.6	− 0.8	− 14.4
	16	− 13.0	0.0	− 13.0
	19	− 13.8	0.0	− 13.8
	22	− 14.5	− 0.4	− 14.9
Dec. 19	1	− 14.3	− 0.7	− 15.0
	4	− 12.9	− 1.4	− 14.3
	7	− 13.4	− 1.2	− 14.6
	10	− 13.4	− 0.8	− 14.2
	13	− 14.6	− 0.8	− 15.4
	16	− 13.6	− 0.8	− 14.4
	19	− 12.6	− 0.4	− 13.0
	22	− 12.9	0.0	− 12.9
Dec. 20	1	− 13.2	− 1.7	− 14.9
	4	− 12.6	− 1.0	− 13.6
	7	− 12.8	− 1.0	− 13.8
	10	− 12.9	− 0.8	− 13.7
	13	− 12.0	− 1.5	− 13.5
	16	− 13.0	− 1.2	− 14.2
	19	− 12.4	− 2.3	− 14.7
	22	− 12.5	− 1.4	− 13.9
Dec. 21	4	− 13.2	− 1.3	− 14.5
	16	− 12.5	− 0.8	− 13.3
Dec. 22	1	− 13.6	0.0	− 13.6
	4	− 13.9	− 0.8	− 14.7
	7	− 13.1	− 2.5	− 15.6
	10	− 13.8	− 0.8	− 14.6
	13	− 13.9	− 0.9	− 14.8
	16	− 13.5	− 1.2	− 14.7
	19	− 12.4	− 1.3	− 13.7
	22	− 13.3	− 1.3	− 14.6
Dec. 23	1	− 14.4	− 1.6	− 16.0
	4	− 14.7	− 1.0	− 15.7
	7	− 13.6	− 0.7	− 14.3
	10	− 13.7	− 0.3	− 14.0

Date	Time MET	Difference in height h_t	Difference in pressure m_t	Corrected difference in height H_t
		mm	0.1 mb	mm
	13	- 13.6	- 0.6	- 14.2
	16	- 13.7	- 0.3	- 14.0
	19	- 13.0	- 0.7	- 13.7
	22	- 12.3	- 1.0	- 13.3
Dec. 24	1	- 13.0	- 1.1	- 14.1
	4	- 13.2	- 0.9	- 14.1
	7	- 12.9	- 1.4	- 14.3
	10	- 12.5	- 1.2	- 13.7
	Sum	- 723.3	- 68.3	- 791.6
	Mean	- 12.69	- 1.20	H = - 13.88

Therefore the correction in atmospheric pressure thus found was applied. The accuracy was improved by so doing:

uncorrected: standard deviation, single observation 1.40 mm
corrected: ,, ,, ,, ,, 1.07 mm

The corrected difference in height became:

nipple Baarland over nipple De Gr. = 13.9 mms. with a standard deviation 0.14 mm.

The total atmospheric correction amounted to 1.2 mm. The difference in height stated above had still to be corrected for the differences in reading of the top and bottom of the meniscus (see 3, 2 under b). $H = 13.9 - 2.6 = 11.3$ mm, standard deviation 0.2 mm.

No corrections needed to be made for the temperatures read on the bottom of the Westerschelde (see table III).

Three factors were favourable in the case of these temperatures.

a. By reason of the great difference in the tide and the swiftness of the current the water in the Westerschelde is mixed in such a way as to make the temperature nearly independent of the depth.

b. In winter the lack of any considerable heating results in only slight differences in temperature between the surface water and that in the depths.

c. Seeing that the temperature was in the neighbourhood of 4° C there was a relatively small difference of density in relation to the differences in temperature.

3, 4. **Result of the measuring of the circuit.**

At the beginning of the third geodetic levelling in 1951 and 1952 measurements were made in W.Brabant and Zeeland. Seeing that these had not yet been linked up

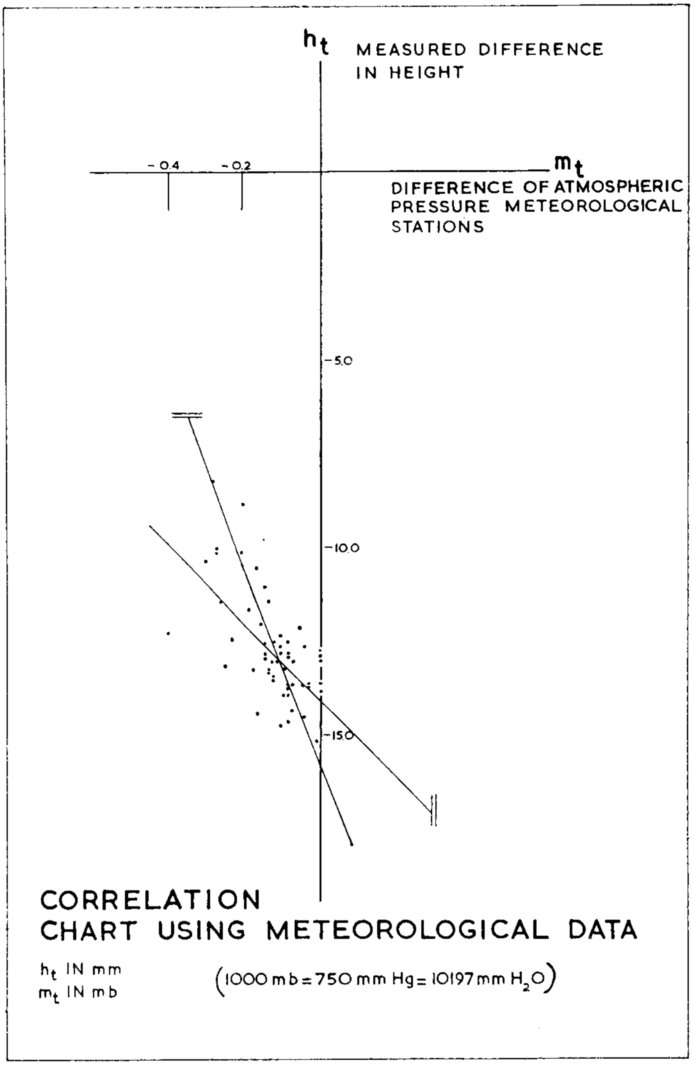


Fig. 10

with the Ordnance Datum at Amsterdam, provisionary heights were used taking the height of the underground bench-mark Gilze Rijen as a starting point.

The circuit closed by the hydrostatic levelling consists of the following sections:

Nipple Baarland — Kapelle — Woensdrecht,
Woensdrecht — underground bench-mark Ossendrecht,
Underground bench-mark Ossendrecht — underground bench-mark Nieuw Namen, (Belgian precision levelling)

Underground bench-mark Nieuw Namen — nipple De Griete.

The total length of the circuit is about 120 kms.
The result was finally:

precise levelling over Antwerp	— 0.0110 m.
hydrostatic levelling	+ 0.0113 m.
	0.0003 m.

From this it follows that the misclosure of the circuit amounts to 0.3 mm. The standard-deviation of the precise-levelling over Antwerp may be estimated at $\sqrt{120} \times 0.6$ mm $= 6.6$ mm.

TABLE III

Measurements of temperature on the bottom of the Westerschelde

Date	Time	Depth in metres	Bottom temperature in °C.	Place (Always sailed in the direction NS).
Dec. 22, 1952	14.18	11.0	4.00	Middelgat (main channel north of the sand-bank)
L.W.	14.34	14.5	4.07	
	14.45	6.0	4.00	
	15.10	5.0	2.83	gat van Ossenisse (channel south of the sand-bank)
	15.25	15.5	2.84	
	15.40	18.0	2.84	
Jan. 30, 1953	9.20	9.0	2.61	Middelgat
L.W.	9.35	12.5	2.61	
	9.50	13.5	2.61	
	10.05	3.5	2.63	
	10.25	6.5	2.64	gat van Ossenisse
	10.40	16.5	2.63	
	11.00	19.0	2.65	
H.W.	15.10	13.0	2.88	Middelgat
	15.23	18.0	2.90	
	15.40	18.5	2.91	
	15.55	14.0	2.93	
	16.15	12.0	2.90	gat van Ossenisse
	16.30	21.5	2.90	

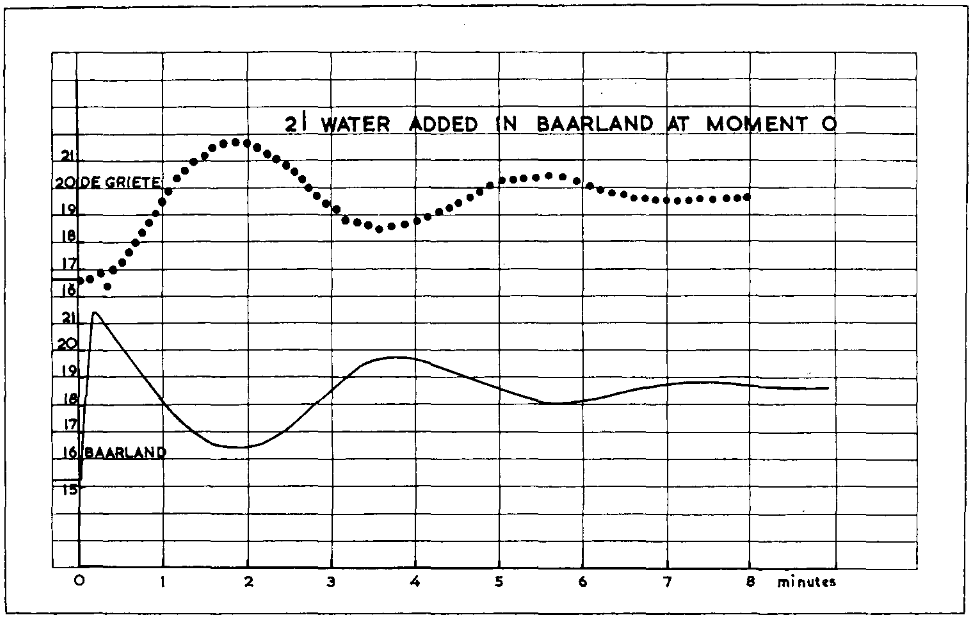
4. SOME TESTS AND CONSIDERATIONS

4, 1. The addition of water.

In order to check whether the pipe was free of air, on Dec. 23 rd, 2 litres of water were poured quickly into the stand-pipe at Baarland and the movements of the water-levels at Baarland and De Griete were watched.

With the help of the electric bell the start of the action of the pouring-in of the water was notified to the observers in De Griete.

The whole mass of the water in the pipe started oscillating with a period of about 220 secs. and an amplitude which decreased rather quickly (fig. 11).



Theoretically the period of oscillation must be equal to:

$$T = \pi \frac{S}{D} \sqrt{\frac{2.l}{g}} = 192 \text{ secs.}$$

in which l = length of water mass in the pipe,
g = acceleration of the gravitational force,
S = diameter of the stand-pipe (20 cm),
D = diameter of gas-main (9,6 cm).

The agreement was not precise because in the formula mentioned no account was taken of friction.

The rises eventually established tally well with those calculated:

	established	*calculated*
Baarland	32.1 mm.	31.68 mm.
De Griete	31.4 mm.	31.68 mm.
Sum	63.5 mm,	63.4 mm.

so that, hereby, it was once again confirmed that the pipe was free of air.

4, 2. The non-stop observations of atmospheric pressure.

On Dec: 22nd the atmospheric pressure was read on the Askania microbarometers every minute from 3 p.m. to 4 p.m. and from 5.33 p.m. to 6.35 p.m. on both banks simultaneously (fig. 12). It appeared that during these $3\frac{1}{2}$ hours the difference in atmospheric pressure decreased by nearly 0.5 mm. mercury without having had any great influence on the difference in height of the water-surfaces. Theoretically the difference in the change of height of the water-surfaces should be: $0.5 \times 13.6 = 6.8$ mm water, whereas only 0.7 mm was recorded as the difference of height of these surfaces. There may be two causes of this:

a. The registering apparatus in the gauge follows (e.g. by reason of friction) only partially the movement of the water-surfaces.
b. One of the two microbarometers did not always quite keep up with the rapid change of atmospheric pressure and lagged behind as it were. (Possibly the instrument at De Griete?)

When we examine the results of the correction of the atmospheric pressure this second cause seems likely.

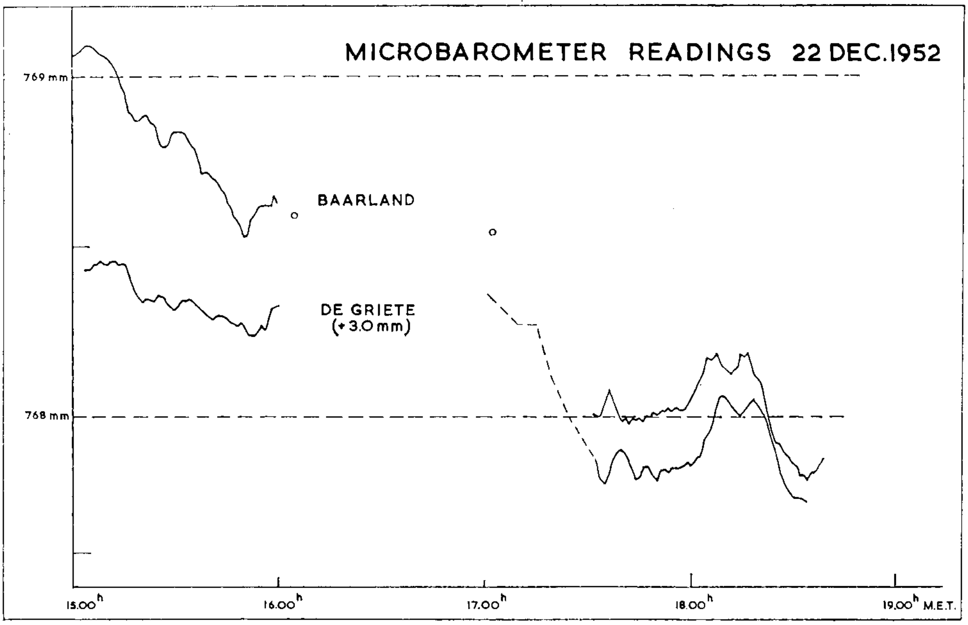


Fig. 12

4, 3. Periodic fluctuations in the water-level.

The violent fluctuations in the water-level in the stand-pipes (Par. 3.2b.) which occurred now and then soon appeared to be but periodic phenomena.

As a result of the registering of the water-level it could be established that the phenomenon occurred, on an average, about 1 hour before High Water and its period varied from 0 to 2 hours.

The maximum amplitude for the gauge was 4 cms, on the graph only 1.8 mm was shown as the maximum, the period of vibration T was only about 1 sec.

In spite of repeated attempts it was not possible to ascertain whether the vibrations occurring on the two banks were simultaneous or if they differed by a phase of $\frac{1}{2}$ T. This was chiefly due to the short period of vibration: it was impossible to transfer the time signal with the desired accuracy. The vibration of the water-levels probably start when the pipe in the Westerschelde begins to vibrate at a certain speed of the tide. The maximum tidal velocity occurred indeed at about 1 hr. before High Water, and according to the report of the commission appointed for the study of the underwater gasmains in the Westerschelde vibrations with a period of about 1 sec. are to be expected then.

4, 4. Change of volume in relation to the tide.

From the graphs of the water-level in the stand-pipes an indication could be found that the heights of the water in the pipe would be dependent on the tide in the Wester-

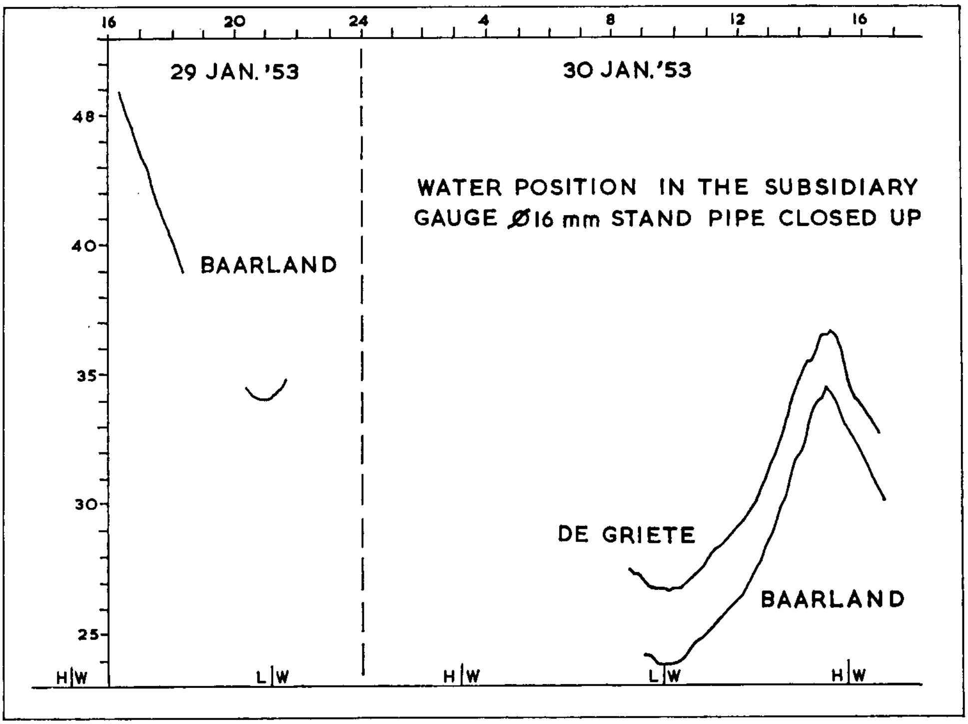


Fig. 13

schelde. In order to confirm this supposition the broad stand-pipe at both observation-stations was closed by welding; because only the small glass-tube served as a stand-pipe a decrease of volume of the whole pipe could be read with greater precision.

On the 29th of Jan. '53 this gauge was read every 5 mins. in Baarland. The water-level fell 12 cms. between 4.00 p.m. and 9.00 p.m. and after that it began to rise once more. The time forecasted for Low Water at Terneuzen was 8.48 p.m. so that L.W. in the Westerschelde tallied very well with the lowest position of water in the pipe. On Jan 30th. the gauge was read every ten minutes between 9.00 a.m. and 4.30 p.m. at both Baarland and De Griete while at the same time the temperatures at the bottom of the Westerschelde were measured during the turn of the tide at both L.W. and H.W. (fig. 13).

The rise from L.W. to H.W. (in the gauge) when totalled amounts to 20.9 cms., this is a volume of $20.9 \times (1.6)^2 \times {}^{16}/_{4} = 42$ cubic cms.

When the decrease in volume resulting from increase of pressure in the steel pipe due to the tidal rise is calculated, it is found that the apparent increase of volume of the water must be this amount:

$$\frac{5.67}{3.33} \frac{P.D.}{2d.E} I = 100 \text{ cubic cms.}$$

P = increase of pressure HW — LW = 0.46 kg/square cm.,
D = external diameter of the pipe = 10.8 cm,
d = thickness of the wall of the pipe = 0.6 cm.,
E = modulus of elasticity of steel, = 2.150.000 kg/sq.cm
I = contents of the whole pipe = 30.400.000 cubic cm.,

Owing to the influence of temperature there will also be a change of volume the size of which is difficult to define because the pipe is insulated by a thick layer of mastic. It is improbable that the contents of the pipe react quickly to temperature changes outside it; for this reason measurements of temperature over a longer period of time would have been necessary to define this volume change.

Because the flood of Febr. 1st. 1953 totally destroyed the observation post at De Griete and heavily damaged that at Baarland (this latter, moreover could not be reached by reason of the damages on the dykes) the observation programme was suddenly put to an end.

4, 5. Considerations of the above facts.

The experiences gained during this river-crossing are of great importance for similar measurement work in the future. The greatest difficulty lay in the preparatory work, namely, in freeing the pipe from air. During the hydrostatic levellings done in Germany in 1952 the same problem arose.

In the Danish publications these difficulties are not mentioned; apparently the system used there was adequate.

Once the pipe is free of air, however, the measurement of the river-crossing is a simple matter. The apparatuses used, in the case of the Westerschelde, for fixing the correction of atmospheric pressure proved to be insufficient. To do the work with, for instance, Vaisälä-statoscopes probably would give a better result. In Denmark excellent results were obtained with similar instruments. The measurement of water temperatures at the sea bottom is not so important in the Dutch estuaries with their strong currents; the question remains, however, whether this conclusion is also valid for the summer season.

The use of automatic gauges is not necessary when the observation programme is limited to a few days which, as experience shows, is very well possible.

The observation programme, however, should not be restricted too much since measurements taken on days with differing structures of the field of atmospheric pressure give a good insight into the reliability of the corrections necessary for atmospheric pressure.

The possibility should be thoroughly considered that with certain speeds of the current in the estuary that is to be crossed, violent vibrations in the gauges may occur. These fluctuations were also recorded in the German levellings in 1952.

Finally, a good telephonic connection between the two observation parts must be considered essential.

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Thanks to their valuable co-operation the hydrostatic levelling could be achieved.

Postscript

Since this report was written there has been much further activity in the field of hydrographic levelling in the Netherlands. Measuring posts on poles were placed in the sea at a considerable distance from the coast. Levellings across sea arms in northwest and southwest Holland (Frisian islands and islands of Zeeland) have been or are being carried out; their results will be discussed perhaps in a later report of this series.

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