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The Assimilation of Altimetric Data into the Barotropic Mode of a Rigid Lid Ocean Model

E. DELEERSNIJDER
G. Lemaître Institute of Astronomy and Geophysics
Catholic University of Louvain
2 Chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium
ericd@astr.ucl.ac.be

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Abstract—Some aspects of the computation of the barotropic mode of a primitive equations, rigid lid, global ocean model are reviewed in the perspective of data assimilation. It is shown that "blending" the surface pressure predicted by the model and that obtained from altimetry is impracticable, since mass conservation must be enforced. By relaxing the momentum equation, however, it is possible to take the measurements into account. A variational method, requiring that the perturbation to the momentum equation be minimal, is suggested. A Poisson equation is obtained for the streamfunction—allowing evaluation of the transport—for which a noniterative solution method is derived. The difference between the curl of the assimilated transport and that ensuing from the model unconstrained by data is investigated, pointing to the role of the bathymetry and that of the "implicitness" in the evaluation of the Coriolis acceleration. Finally, implications for altimetric data assimilation of the elliptic nature of the rigid lid equations are presented, and a comparison with the hyperbolic equations of a free surface model is outlined.

Keywords—Ocean modelling, Altimetry, Data assimilation, Rigid lid, Barotropic mode.

1. INTRODUCTION

For altimeter data assimilation, more benefit is generally expected from converting elevation data into density data pertaining to the interior of the ocean than from directly constraining the evolution of the barotropic mode with altimetric measurements [1–2]. This does not imply, however, that the assimilation of surface elevation data into the barotropic mode of an ocean model should not be attempted. Consider, for example, a World Ocean model: the flow in the Antarctic Circumpolar Current (ACC) is much less "baroclinic" than anywhere else. Not surprisingly, it is only in this area that the "barotropic component" of the ocean surface elevation is significant relative to its "baroclinic" counterpart [3]. In view of the importance of the ACC, it is clear that assimilation techniques for combining altimetric data and the barotropic mode dynamics must not be ignored.

In a recent article, Pinardi *et al.* [3] reviewed, in the perspective of altimetric data assimilation, several aspects of the computation of the barotropic mode of a primitive equations, rigid lid,

Mailing address: Unité ASTR à l'UCL, 2, Chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium.
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global ocean model. They considered a simple assimilation procedure consisting of "blending" the altimetric measurement η_d and the elevation η_m predicted by the model when unconstrained by data. They suggested forcing the barotropic mode momentum equation by an appropriate linear combination of η_d and η_m,

$$\eta = \alpha\eta_d + (1-\alpha)\eta_m, \qquad 0 \leq \alpha \leq 1, \tag{1}$$

in the hope that the resulting barotropic velocity field would be closer to reality than that obtained by ignoring data, i.e., by setting $\alpha = 0$ in (1).

As shown by Pinardi *et al.* [3], the method above is ill-designed, for the conservation of mass precludes any modification of the ocean surface elevation in a rigid lid model. This conclusion, though correct, stems from two rather restrictive hypotheses. First, it is assumed that the barotropic continuity equation, stating that the transport must be divergenceless, has to be rigorously enforced. Second, the barotropic momentum equation is considered exact. Lifting at least one of those hypotheses would permit implementing an assimilation method based on the simple blending formula (1). We feel that it might not be safe to violate the mass conservation—even if an appropriate penalty function is introduced in the scope of a variational approach. Thus, we would suggest disposing with the second condition. Accordingly, a variational method is examined herein, the essence of which is to introduce minimal perturbation in the momentum equation of the barotropic mode.

Although our conclusions somewhat contradict those of Pinardi *et al.* [3], they do not shed criticism on their work, just because our standpoint is significantly different from theirs. On the other hand, our discussion is mostly speculative, since no actual application is presented. As a result, the aim of this note is solely to prompt discussion and, possibly, controversy on the way to assimilate altimeter data into the barotropic mode of a rigid lid ocean model.

Before introducing our variational method, it is necessary to briefly recall how the equations of the barotropic mode are established and why Pinardi *et al.* [3] found that an assimilation procedure resting on (1) is impracticable.

2. BAROTROPIC MODE

As in most ocean models, the Boussinesq approximation and the hydrostatic equilibrium are assumed valid.

The hydrostatic equilibrium equation reads

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2}$$

where p, z, ρ, and g denote the pressure, the vertical coordinate (pointing upwards), the density, and the gravitational acceleration, respectively. Hence, at time t and location $(\mathbf{x}, z)$, the pressure is given by

$$p(t,\mathbf{x},z) = p_a(t,\mathbf{x}) + g\int_z^\eta \rho(t,\mathbf{x},z)\,dz, \qquad -h \leq z \leq \eta, \tag{3}$$

where the sea surface elevation η is positive when the ocean surface is above the reference level $z = 0$; p_a represents the atmospheric pressure at sea level. Since η is usually much smaller than the ocean depth h, and since $|(\rho - \rho_0)/\rho_0| \ll 1 - \rho_0$ being an appropriate reference value of ρ—, it is clear that expression (3) may be approximated by

$$p(t,\mathbf{x},z) = p_a(t,\mathbf{x}) + \rho_0 g\eta(t,\mathbf{x}) + g\int_z^0 \rho(t,\mathbf{x},z)\,dz. \tag{4}$$

The mass conservation over a water column requires

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\mathbf{U} = 0, \tag{5}$$

where ∇ is the horizontal "gradient operator" and $\mathbf{U}$ denotes the transport, i.e., the depth-integral of the horizontal velocity. If the first term in the left-hand side of (5) is small, the rigid lid approximation may be called on, so that (5) simplifies to

$$\nabla \cdot \mathbf{U} = 0. \tag{6}$$

One may then consider that an impermeable, flat and rigid lid is placed at $z = 0$. If $p_s(t, \mathbf{x})$ represents the pressure exerted by the flow on this fixed boundary, the integration of (2) along the vertical coordinate yields

$$p(t, \mathbf{x}, z) = p_s(t, \mathbf{x}) + g \int_z^0 \rho(t, \mathbf{x}, z)\, dz, \qquad -h \leq z \leq 0. \tag{7}$$

If the pressure computed in formula (4) is equivalent to that obtained in the framework of the rigid lid approximation, then

$$p_s = p_a + \rho_0 g \eta, \tag{8}$$

which allows linking the altimetric data and the surface pressure.

The density ρ is computed from the equation of state once the temperature and the salinity are known. The latter quantities are easily determined from the relevant evolution equations. There is, however, no evolution equation available to determine p_s. Thus, for the moment, the pressure as expressed in relation (7) is defined up to a function of t and $\mathbf{x}$.

Since p_s is independent of the vertical coordinate, it is clear that it should be determined with the help of the barotropic mode equations.

The horizontal pressure gradient force reads

$$-\frac{1}{\rho_0} \nabla p = -\frac{1}{\rho_0} \nabla p_s - \frac{g}{\rho_0} \int_z^0 \nabla \rho(t, \mathbf{x}, z)\, dz, \tag{9}$$

which allows the barotropic momentum equation to be written as

$$\frac{\partial \mathbf{U}}{\partial t} + f \mathbf{e}_z \times \mathbf{U} = -\frac{h}{\rho_0} \nabla p_s + \mathbf{E}, \tag{10}$$

where f and $\mathbf{e}_z$ represent the Coriolis factor and the vertical unit vector, respectively; $\mathbf{E}$ collects the depth-integral of the last term of (9), as well as advection, diffusion and bottom/surface stress terms. Taking the divergence of (10), using (6), one obtains

$$\nabla \cdot \left[\frac{h}{\rho_0} \nabla p_s \right] = \nabla \cdot (\mathbf{E} - f \mathbf{e}_z \times \mathbf{U}). \tag{11}$$

The latter formula is a Poisson equation for the surface pressure p_s, pointing to the elliptic nature of the barotropic mode equations of a rigid lid model.

As the present discussion is carried out at the conceptual level, it is not necessary to consider the equations and their numerical discretization in their full complexity. However, some technical details must be included, otherwise our work would be misleading and pointless.

3. SURFACE PRESSURE COMPUTATION

As Pinardi *et al.* [3], we envisage the modelling of the World Ocean. Therefore, the computational domain is assumed to be a closed basin, possibly containing I islands (Figure 1). It is,

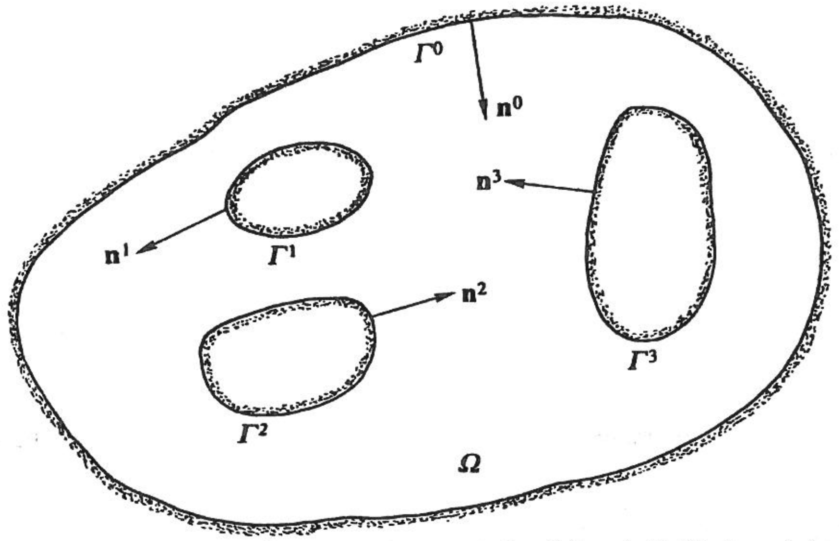

Figure 1. Schematic representation of the computational domain Ω. The boundaries $\Gamma^i (0 \leq i \leq I)$ are impermeable and $\mathbf{n}^i$ denote their normal unit vectors. Here, it is assumed that there are 3 islands in Ω, i.e., $I = 3$.

however, not essential to use spherical coordinates. Our ocean is thus assumed to be located on a flat Earth, and is described in the Cartesian coordinate system.

As emphasized by several authors, it is important that the Coriolis acceleration be discretized with some degree of "implicitness" [4–6]. Accordingly, we consider the following time discretization of the barotropic mode equations (6) and (10):

$$\nabla \cdot \mathbf{U}^{n+1} = 0, \tag{12}$$

$$\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} + \frac{\omega}{\Delta t}\mathbf{e}_z \times \mathbf{U}^{n+1} + \frac{\tilde{\omega}}{\Delta t}\mathbf{e}_z \times \mathbf{U}^n = -\frac{h}{\rho_0}\nabla p_s^{n+1} + \mathbf{E}^n, \tag{13}$$

with $\omega = \beta f \Delta t (0 \leq \beta \leq 1)$ and $\omega + \tilde{\omega} = f\Delta t$; superscript "$n$" is associated, in the usual way, with the time discretization. The "implicitness" factor ω is assumed constant in space.

Let

$$\mathbf{F}^n = \mathbf{U}^n - \tilde{\omega}\mathbf{e}_z \times \mathbf{U}^n + \Delta t \mathbf{E}^n. \tag{14}$$

Substituting the definition (14) into the momentum equation (13) yields

$$\left(1 + \omega^2\right)\mathbf{U}^{n+1} = -\frac{h\Delta t}{\rho_0}\left(1 - \omega\mathbf{e}_z\times\right)\nabla p_s^{n+1} + \left(1 - \omega\mathbf{e}_z\times\right)\mathbf{F}^n. \tag{15}$$

Taking the divergence of the discretized equation (15) using (12), one obtains the discretized counterpart of (11) [6]

$$\nabla \cdot \left[h\left(1 - \omega\mathbf{e}_z\times\right)\nabla p_s^{n+1}\right] = \frac{\rho_0}{\Delta t}\nabla \cdot \left[\left(1 - \omega\mathbf{e}_z\times\right)\mathbf{F}^n\right]. \tag{16}$$

A straightforward discretization of (11) would have led to a form very different from (16). Hence, the need to consider some of the numerical technicalities—as stated above.

The left-hand member of (16) may be developed as follows:

$$\nabla \cdot \left[h\left(1 - \omega\mathbf{e}_z\times\right)\nabla p_s^{n+1}\right] = h\nabla^2 p_s^{n+1} + \nabla h \cdot \left(1 - \omega\mathbf{e}_z\times\right)\nabla p_s^{n+1}, \tag{17}$$

clearly showing that (16) is elliptic. Thus, the time discretization used herein preserves the elliptic nature of the problem.

The computational domain only has impermeable boundaries. Thus, if $\mathbf{n}^i$ is the normal to the boundary segment $\Gamma^i (0 \leq i \leq I)$ (Figure 1), with the help of (15), the impermeability conditions $\mathbf{n}^i \cdot \mathbf{U}^{n+1} = 0$ may be written as

$$\mathbf{n}^i \cdot \left[-\frac{h\Delta t}{\rho_0} \left(1 - \omega \mathbf{e}_z \times\right) \nabla p_s^{n+1} + \left(1 - \omega \mathbf{e}_z \times\right) \mathbf{F}^n \right] = 0, \qquad 0 \leq i \leq I, \tag{18}$$

which are Neumann boundary conditions for p_s^{n+1}.

It must be stressed that relations (18) are not additional boundary conditions for the surface pressure, for they stem from the need to satisfy the impermeability of the boundaries of the domain of interest [7,8].

Equation (16) together with the boundary conditions (18) form an elliptic partial differential problem, the solution of which is unique—up to an additive constant, which has actually no practical importance. The unicity of the surface pressure implies that a simple "blending" assimilation technique cannot work. This is readily seen. If $p_{s,m}^{n+1}$ represents the surface pressure computed by the model as described above, and if $p_{s,d}^{n+1}$ is the surface pressure derived from altimeter data according to (8), one may re-define p_s^{n+1}—in agreement with (1)—as

$$p_s^{n+1} = \alpha p_{s,d}^{n+1} + (1 - \alpha) p_{s,m}^{n+1}. \tag{19}$$

It is then tempting to introduce the above value of the surface pressure into momentum equation (13) so as to update the transport. But, doing so, one would obtain a field of $\mathbf{U}^{n+1}$ having nonzero divergence, since it is only when $p_s^{n+1} = p_{s,m}^{n+1}$, i.e., for $\alpha = 0$, that$\nabla \cdot \mathbf{U}^{n+1} = 0$.

Since we considered it inappropriate to relax the continuity equation (12), it is only by relaxing the momentum equation (13) itself that a simple procedure for using the "blended" surface pressure (19) may be set up. Such a method is described below.

4. A VARIATIONAL METHOD

Even if the formulation of the barotropic momentum equation (10) is deemed satisfactory (i.e., if the parameterizations are well suited to the flow under study), its discretized version (13), or equivalently (15), encompasses several errors. There are errors stemming from the space and time discretization. In addition, it must be realized that none of the model variable at instant $n\Delta t, \mathbf{U}^n, \rho^n$, etc., is exact. Thus, to find the *in situ* value of $\mathbf{U}^{n+1}$ with the help of (15), a term corresponding to all these errors should be added to (13). As a result, the exact field of $\mathbf{U}^{n+1}$ would be given by

$$\left(1 + \omega^2\right) \mathbf{U}^{n+1} = -\frac{h\nabla t}{\rho_0} \left(1 - \omega \mathbf{e}_z \times\right) \nabla p_s^{n+1} + \left(1 - \omega \mathbf{e}_z \times\right) \mathbf{F}^n + \mathbf{r}, \tag{20}$$

where $\mathbf{r}$ represents the error affecting the right-hand member of (15). There is no way to accurately determine $\mathbf{r}$, except in very limited areas of space-time where *in situ* measurements of $\mathbf{U}^{n+1}$ are available.

We are, however, not at a dead end. Instead, we adopt the following pragmatic approach. We suggest taking advantage of the fact that perturbations to (15) must be taken into account in order to improve the accuracy of the predicted field of transport. Accordingly, in (20), we consider that p_s^{n+1} is actually the "blended" pressure (19), in the hope that the pressure term, and ultimately the modelled transport, be more accurate—than it would be by using $p_{s,m}^{n+1}$ alone. This way, with $\mathbf{r} = 0$, $\mathbf{U}^{n+1}$ would have nonzero divergence, as already shown. We then suggest finding the smallest possible perturbation $\mathbf{r}$ ensuring that $\nabla \cdot \mathbf{U}^{n+1} = 0$. With such a practice, $\mathbf{r}$ will not be equal to the error affecting the right-hand member of (15). However, since it is generally impossible to evaluate this error, a reasonable option may be to call on the smallest possible perturbation to (15). To do so, we must define an appropriate measure of $\mathbf{r}$ over the computational domain Ω. It is suggested seeking the minimum of the following functional

$$J = \int_{\Omega} \frac{|\mathbf{r}|^2}{(1+\omega^2)^2}\, d\Omega \tag{21}$$

under the constraint that $\nabla \cdot \mathbf{U}^{n+1} = 0$. To make sure that $\mathbf{U}^{n+1}$ is indeed divergenceless, it is convenient to define it with the help of a streamfunction ψ, i.e.,

$$\mathbf{U}^{n+1} = \mathbf{e}_z \times \nabla\psi. \tag{22}$$

Let $\mathbf{U}^*$ be defined as

$$\mathbf{U}^* = \frac{-(h\Delta t)/(\rho_0)\,(1-\omega\mathbf{e}_z\times)\,\nabla p_s^{n+1} + (1-\omega\mathbf{e}_z\times)\,\mathbf{F}^n}{1+\omega^2}. \tag{23}$$

Combining (20)–(23), one has

$$J = \int_{\Omega} \left|\mathbf{U}^{n+1} - \mathbf{U}^*\right|^2 d\Omega = \int_{\Omega} |\nabla \times \psi\mathbf{e}_z + \mathbf{U}^*|^2\, d\Omega. \tag{24}$$

The minimum of J over ψ thus corresponds to the minimum of a global, quadratic measure of $\mathbf{r}$. Another interpretation is however possible. Indeed, $\mathbf{U}^*$ may be regarded as the transport one would obtain by using the "blended" pressure and without considering any perturbation to (15), i.e., by setting $\mathbf{r} = 0$. The transport $\mathbf{U}^{n+1}$ achieving the minimum of J is thus the divergenceless transport that is closest to $\mathbf{U}^*$.

Expressing that the first order variation of J must be zero, we obtain the following Euler-Lagrange equation

$$\nabla^2\psi = \mathbf{e}_z \cdot (\nabla \times \mathbf{U}^*) \qquad \text{in } \Omega, \tag{25}$$

It is required that the water flux across the boundaries be zero. Accordingly, the streamfunction is ascribed to constant values ψ^i on the boundaries, i.e.,

$$\psi = \psi^i, \qquad \text{on } \Gamma^i (0 \le i \le I), \tag{26}$$

where the constants ψ^i are yet to be determined. Equation (25) is of elliptic nature, and thus only allows one boundary condition on Γ^i. As a result, it is no longer possible to use no-slip boundary conditions as is often done in B-grid ocean models.

It is easily shown that $\mathbf{e}_z \cdot \left(\nabla \times \mathbf{U}^{n+1}\right) = \nabla^2\psi$. Therefore, (25) simply states that $\mathbf{U}^{n+1}$ and $\mathbf{U}^*$ have the same curl.

Since adding a constant to the streamfunction ψ has no dynamical effect, one may set

$$\psi^0 = 0, \tag{27}$$

without any loss of generality.

5. DISCUSSION

If there is no island in the domain of interest ($I = 0$), it is sufficient to solve (25) subject to the boundary condition (27).

If there is at least one island ($I \ge 1$), there are I constants to be determined in such a way that J be minimum. A brute force treatment of this minimum problem is likely to require an iterative method in which the Poisson equation (25) will have to be solved several times. It is however possible to avoid having recourse to an iterative technique.

We write ψ as (see [9])

$$\psi(\mathbf{x}) = \zeta(\mathbf{x}) + \sum_{i=1}^{I} \phi_i(\mathbf{x})\psi^i, \tag{28}$$

where ζ is the solution to the following elliptic problem

$$\nabla^2\zeta = \mathbf{e}_z \cdot (\nabla \times \mathbf{U}^*), \qquad \text{in } \Omega, \tag{29}$$

$$\zeta = 0, \qquad \text{on } \Gamma^i (0 \le i \le I), \tag{30}$$

and where the functions ϕ_i verify, for $1 \le i \le I$,

$$\nabla^2\phi_i = 0, \qquad \text{in } \Omega, \tag{31}$$

$$\phi_i = \delta_{ik}, \qquad \text{on } \Gamma^k (0 \le k \le I), \tag{32}$$

with $\delta_{ik} = 1$ if $i = k$, and $\delta_{ik} = 0$, otherwise. It is readily understood that definition (28) together with the problems (29)–(30) and (31)–(32) lead to a value of the streamfunction that is equal to that ensuing from the original problems (25)–(26).

Introducing (28) into (24), expressing that J must be minimum by requiring that

$$\frac{\partial J}{\partial \psi^i} = 0, \qquad 1 \le i \le I, \tag{33}$$

one obtains an algebraic system of I linear equations

$$\sum_{k=1}^{I} \left[\int_\Omega (\nabla \times \phi_k \mathbf{e}_z) \cdot (\nabla \times \phi_i \mathbf{e}_z)\, d\Omega \right] \psi^k = -\int_\Omega (\nabla \times \zeta\mathbf{e}_z + \mathbf{U}^*) \cdot (\nabla \times \phi_i \mathbf{e}_z)\, d\Omega, \qquad 1 \le i \le I. \tag{34}$$

Since the functions ϕ_i and the coefficients of ψ^k in the left-hand side of system (34) are independent of $\mathbf{U}^*$, they may be calculated once and for all. As a consequence, at each time step, it is necessary to solve only one Poisson equation—for determining ζ.

In its present state of elaboration, the noniterative method may present a serious drawback. It is necessary to store I two-dimensional arrays containing the values of ϕ_i at all nodes of the numerical grid, which may demand a prohibitively large amount of memory in a fine resolution World Ocean model in which there can be hundreds, or even thousands, of islands—i.e., I may be of order 10^2 or 10^3. Fortunately, the right-hand member of (34) may be transformed into an expression independent of ϕ_i (see [9]).

Upon setting

$$\mathbf{G} = \nabla \times \zeta\mathbf{e}_z + \mathbf{U}^*, \tag{35}$$

one may re-write the right-hand side of (34) as

$$-\int_\Omega (\nabla \times \zeta\mathbf{e}_z + \mathbf{U}^*) \cdot (\nabla \times \phi_i \mathbf{e}_z)\, d\Omega = \int_\Omega [\nabla \cdot (\mathbf{G} \times \phi_i \mathbf{e}_z) + \phi_i \mathbf{e}_z \cdot (\nabla \times \mathbf{G})]\, d\Omega. \tag{36}$$

Equation (29) implies that $\nabla \times \mathbf{G} = 0$. Thus, with the help of Gauss' theorem, (36) simplifies to

$$-\int_\Omega (\nabla \times \zeta\mathbf{e}_z + \mathbf{U}^*) \cdot (\nabla \times \phi_i \mathbf{e}_z)\, d\Omega = \sum_{k=1}^{I} \int_{\Gamma^k} \mathbf{n}^k \cdot (\mathbf{G} \times \phi_i \mathbf{e}_z)\, d\Gamma^k. \tag{37}$$

Since, by definition, $\phi_i = \delta_{ik}$ on Γ^k, one finally obtains

$$-\int_{\Omega} (\nabla \times \zeta \mathbf{e}_z + \mathbf{U}^*) \cdot (\nabla \times \phi_i \mathbf{e}_z) \, d\Omega = \int_{\Gamma^i} \mathbf{n}^i \cdot (\mathbf{G} \times \mathbf{e}_z) \, d\Gamma^i. \tag{38}$$

Hence, the system may now be cast into a form unlikely to cause storage problems, i.e.,

$$\sum_{k=1}^{I} \left[\int_{\Omega} (\nabla \times \phi_k \mathbf{e}_z) \cdot (\nabla \times \phi_i \mathbf{e}_z) \, d\Omega \right] \psi^k = \int_{\Gamma^i} \mathbf{n}^i \cdot [(\nabla \times \zeta \mathbf{e}_z + \mathbf{U}^*) \times \mathbf{e}_z] \, d\Gamma^i. \tag{39}$$

It is now attempted to assess to what extent the transport $\mathbf{U}^{n+1}$ ensuing from our assimilation method is different from that the model unconstrained by data would predict, which we denote $\mathbf{U}_m^{n+1}$.

For clarity, we write down $\mathbf{U}_m^{n+1}$ and $\mathbf{U}^*$ as

$$\mathbf{U}_m^{n+1} = ah\,(1 - \omega \mathbf{e}_z \times)\, \nabla p_{s,m}^{n+1} + b\,(1 - \omega \mathbf{e}_z \times)\, \mathbf{F}^n, \tag{40}$$

$$\mathbf{U}^* = ah\,(1 - \omega \mathbf{e}_z \times)\, \nabla \left[\alpha p_{s,d}^{n+1} + (1-\alpha) p_{s,m}^{n+1} \right] + b\,(1 - \omega \mathbf{e}_z \times)\, \mathbf{F}^n, \tag{41}$$

with $a = -\Delta t / \left[\rho_0 \left(1 + \omega^2\right)\right]$ and $b = 1/\left(1 + \omega^2\right)$, so that

$$\mathbf{U}^* = \mathbf{U}_m^{n+1} + \alpha a h\,(1 - \omega \mathbf{e}_z \times)\, \nabla \left(p_{s,d}^{n+1} - p_{s,m}^{n+1} \right). \tag{42}$$

Since the curl of $\mathbf{U}^{n+1}$ and that of $\mathbf{U}^*$ are equal, relation (42) leads to

$$\begin{aligned} \nabla \times \mathbf{U}^{n+1} - \nabla \times \mathbf{U}_m^{n+1} = {} & \alpha a \nabla h \times \left[(1 - \omega \mathbf{e}_z \times)\, \nabla \left(p_{s,d}^{n+1} - p_{s,m}^{n+1} \right) \right] \\ & - \alpha a h \omega \nabla^2 \left(p_{s,d}^{n+1} - p_{s,m}^{n+1} \right) \mathbf{e}_z, \end{aligned} \tag{43}$$

which clearly shows that, in general, $\mathbf{U}^{n+1}$ is different from $\mathbf{U}_m^{n+1}$, which is the least to be expected from the present assimilation method! It is thus through the interaction of the depth gradient and the pressure gradient and through the implicit treatment of the Coriolis acceleration that the "blending" of the model and surface pressure induces modifications into the curl of the transport.

If the right-hand side of (40) is zero at every location in Ω—which is the case when the bottom is flat and when $\omega = 0$, for example—then $\mathbf{U}^{n+1}$ may still be different from $\mathbf{U}_m^{n+1}$. Since $p_{s,m}^{n+1}$ is determined so that the divergence of $\mathbf{U}_m^{n+1}$ be zero, one may consider that $\mathbf{U}_m^{n+1}$ derives from a streamfunction, i.e.,

$$\mathbf{U}_m^{n+1} = \mathbf{e}_z \times \nabla \psi_m. \tag{44}$$

With this definition, the equality of the curls of $\mathbf{U}^{n+1}$ and $\mathbf{U}_m^{n+1}$ implies

$$\nabla^2 \psi = \nabla^2 \psi_m. \tag{45}$$

If there is no island, the boundary condition applying to ψ and ψ_m are equivalent, i.e., $\psi = \psi_m = 0$ on Γ^0. Therefore, $\psi = \psi_m$ in Ω, so that $\mathbf{U}^{n+1} = \mathbf{U}_m^{n+1}$.

If there is at least one island, there is no reason to believe that ψ and ψ_m are equal on $\Gamma^i (1 \leq i \leq I)$, so that ψ and ψ_m are most probably different in Ω. In this case, $\mathbf{U}^{n+1} - \mathbf{U}_m^{n+1}$ is a divergenceless and irrotational vector field having zero normal component on the boundaries Γ^i.

The present discussion would not be complete without a few words on the fundamental difference in the nature of the free surface and the rigid lid problems.

The barotropic mode equations of the free surface problem, namely (5) and (10), are hyperbolic if the horizontal friction terms are neglected. (Since the latter terms are small and since the boundary layers associated with them are usually disregarded in ocean models, neglecting them is a reasonable assumption in the framework of a conceptual discussion.) According to the hyperbolic nature of the problem, the ocean surface elevation and the two components of the transport may be initialized independently. On the other hand, it is clear that an assimilation method using the "blending" formula (1) may be interpreted as a re-initialization of η in the course of the solution procedure of the model equations. Such a re-initialization is permitted by the mathematical nature of the problem, and is thus unlikely to cause serious difficulties. This does not mean that we believe that this kind of method should be preferred. It is only asserted that this technique does not contradict any fundamental mathematical principle.

By contrast, (6) and (10), which govern the evolution of the rigid lid barotropic mode, are elliptic, so that the influence of perturbations of every variable instantaneously affects the other variables and is immediately felt in the whole domain of interest. This may prevent any independent initialization—or re-initialization—of the variables of the problem. Indeed, for the problem under study, only one variable, the streamfunction from which $\mathbf{U}$ may be derived, may be freely initialized. As we did not consider that data pertaining to ψ were available, it is thus not surprising that the method proposed involves the simultaneous modification of all variables of the problem. It is also little wonder that an elliptic partial differential problem forms the core of our assimilation technique.

6. CONCLUSION

It must be stressed again that the present work, because of its theoretical and speculative nature, cannot give any indication as to the usefulness of the assimilation method proposed. For example, one may argue that the difference between $\mathbf{U}^{n+1}$ and $\mathbf{U}_m^{n+1}$ may be so large that the deep ocean circulation will be severely perturbed, eventually leading to the generation of a significant amount of spurious gravity waves [10].

It is clear that only numerical experiments will reveal the actual potential of our approach.

One advantage may already be highlighted: our method is rather simple, so that it is likely to be much less computer expensive than more sophisticated techniques—which are, however, probably more efficient. Nevertheless, devoting many computer resources to assimilation in the barotropic mode is unlikely to be the best option, just because it is believed that more benefit may be expected from assimilation pertaining to other components of the model. Therefore, a simple and economical method may be sufficient for assimilating altimeter data into the barotropic mode of a primitive equations, rigid lid, global ocean model.

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